Bremsstrahlung in α Decay

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We present the first fully quantum mechanical calculation of photon radiation accompanying charged particle decay from a barrier resonance. The soft-photon limit agrees with the classical results, but differences appear at next-to-leading order. Under the conditions of α decay of heavy nuclei, the main contribution to the photon emission stems from Coulomb acceleration and may be computed analytically. We find only a small contribution from the tunneling wave function under the barrier. [S0031-9007(98)05949-3]

Nuclear fission and α decay are interesting processes that involve both tunneling and the acceleration of charged particles in Coulomb fields. This raises the question of whether the tunneling process affects the bremsstrahlung emission. A semiclassical theory with an affirmative conclusion has been given by Dyakonov and Gornyi in Ref. [1]. In an experiment on the spontaneous fission of ^{252}Cf, Luke *et al.* [2] found a null result and gave an upper limit to the bremsstrahlung rate. In the case of α decay of heavy nuclei, two recent experiments by D'Arrigo *et al.* [3] and Kasagi *et al.* [4] detected accompanying photon radiation. The latter authors claimed to observe interference effects with a tunneling contribution to the bremsstrahlung, interpreting their results in the framework of Ref. [1]. This gives urgency to carry out a full quantum mechanical calculation of the bremsstrahlung. We describe here a calculation with the following assumptions: a single-particle barrier model to describe the α-nucleus wave function, initial state treated as a Gamov state with a complex energy, and perturbative coupling of the photon to the current, taken in the dipole approximation.

The perturbative expression for the photon emission probability during decay may be obtained from Fermi's golden rule

$$w_{i \to f} = \frac{2\pi}{\hbar} \left| \left\langle f \left| \frac{q}{mc} \vec{A} \cdot \vec{p} \right| i \right\rangle \right|^2 \rho_f. \quad (1)$$

In what follows we treat the photon field $\vec{A}$ in the dipole approximation and use the identity $\langle f|\vec{p}|i\rangle = \langle f|[H,\vec{p}]|i\rangle/E_\gamma = i\hbar\langle f|\nabla V|i\rangle/E_\gamma$, where H is the α-particle Hamiltonian with potential V. One obtains

$$\frac{dP}{dE_\gamma} = \frac{4Z_{\text{eff}}^2 e^2}{3m^2 c^3} |\langle \Phi_f|\partial_r V|\Phi_i\rangle|^2 \frac{1}{E_\gamma}. \quad (2)$$

Here dP/dE_γ is the branching ratio to decay with a photon emission, differential in the photon energy E_γ. The wave functions $\Phi_i(r)$ and $\Phi_f(r)$ are the radial wave functions of the initial and final states of the α particle, respectively, with normalization specified below. The effective charge Z_{eff} for dipole transitions is given by $Z_{\text{eff}} = [(A-4)z - 4(Z-2)]/A$ where $z = 2$ is the

charge of the α particle, and Z and A are the charge and mass number of the decaying nucleus, respectively.

We take the potential of the single-particle Hamiltonian as the Coulomb outside a radius r_0 and a constant inside,

$$V(r) = \frac{Zze^2}{r} \Theta(r - r_0) - V_0\Theta(r_0 - r). \quad (3)$$

We will see later that the results are quite insensitive to the choice of parameters V_0 and r_0, provided the decay properties are reproduced. The initial state Φ_i is a resonant state of zero angular momentum normalized to a unit outgoing flux of particles. Its radial wave function is given in terms of F_0 and G_0 Coulomb wave functions by $(\frac{m}{\hbar k})^{1/2}[G_0(\eta, kr) + iF_0(\eta, kr)]/r$ outside r_0 and is proportional to the $j_0(\kappa r)$ spherical Bessel function inside. The Sommerfeld parameter η is given by $\eta = \frac{zZe^2m}{\hbar^2 k}$; it is much larger than one for heavy nuclei. The wave numbers k and κ satisfy $k = \hbar^{-1}\sqrt{2mE_\alpha}$ and $\kappa = \hbar^{-1}\sqrt{2m(E_\alpha + V_0)}$ where E_α is the α-decay energy. Matching the wave functions at $r = r_0$ yields the amplitude of the inner wave function as well as the (complex) energy of the resonant state.

The parameters r_0 and V_0 of our nuclear potential (3) are fixed to reproduce the empirical decay energy E_α and mean life τ of the decay. The mean lifetime depends on the parameters through the equation

$$\frac{2E_\alpha\tau}{\hbar} \approx \frac{kr_0}{2} \frac{G_0^2(\eta, kr_0)}{\sin^2 \kappa r_0} \left(1 - \frac{\sin 2\kappa r_0}{2\kappa r_0}\right) + \int_{kr_0}^{2\eta} d\rho\, G_0^2(\eta, \rho). \quad (4)$$

In Eq. (4) and also in the wave function matching, we neglect terms with F_0 and F_0' which are of order $O(\Delta) \ll 1$ compared to G_0 and G_0'. Here $\Delta = \frac{\hbar}{\tau E_\alpha}$ is a small parameter, and the primes denote derivatives with respect to kr.

As is well known, there are multiple solution sets (r_0, V_0) for a given decay energy and mean life, distinguished by the number of nodes of the inner wave function [5–7]. Typical solution sets for the nuclei of interest are shown in Table I. Our simple model gives reasonable radii

TABLE I. Parameters of the nuclear potential. The table lists the parameters r_0 and V_0 of the nuclear potential (3) that were used in the present calculations. For each listed nucleus, the presented parameters reproduce the experimentally known values of the α-particle energy E_α and the mean lifetime τ.

	^{214}Po	^{210}Po	^{226}Ra
$E/$MeV	7.7	5.3	4.8
$\Delta = \hbar/\tau E_\alpha$	3.7×10^{-19}	7.3×10^{-30}	1.9×10^{-33}
$r_0/$fm	9.19	8.76	9.75
$V_0/$MeV	12.1	16.7	12.9

close to or slightly larger than the nuclear radius for V_0 in the range 0 to 150 MeV [5–7]. The results presented below do not depend on a specific choice of a solution. In the case of vanishing photon energy the parameters drop out; for finite photon energies, the numerical results exhibit no dependence on the choice of parameters.

The continuum final states Φ_f are normalized to give the unit operator when integrated over energy, $\delta(r - r') = \int dE\, \Phi_E(r)\Phi_E(r')$. The radial wave function is a standing wave having the form $(\frac{2m}{\pi \hbar^2 k'})^{1/2}[G_1(\eta', k'r)\sin\alpha + F_1(\eta', k'r)\cos\alpha]/r$ outside r_0 and proportional to the $j_1(\kappa' r)$ spherical Bessel function inside. Primed quantities are defined similar to the corresponding unprimed quantities, but with the energy diminished by the emission of a photon. Matching the wave function at $r = r_0$ yields expressions for $\tan\alpha$ and the amplitude of the inner wave function. The final state is off resonance for almost all final energies $(E_\alpha - E_\gamma)$, and thus $\tan\alpha \sim O(\Delta) \ll 1$; i.e., the final wave function does not penetrate the nucleus significantly and is a true continuum wave function. Outside the Coulomb barrier the wave function is very well approximated by the regular Coulomb wave function only.

The matrix element in Eq. (2) can now be written down. It has a delta function contribution to r_0 and an integral over the derivative of the Coulomb field outside,

$$\langle \Phi_f|\partial_r V|\Phi_i\rangle = \sqrt{\frac{2m^2}{\pi\hbar^3 kk'}}\left\{\left(\frac{zZe^2}{r_0} + V_0\right)[F_1(\eta', k'r_0) + G_1(\eta', k'r_0)\tan\alpha]G_0(\eta, kr_0)\right.$$
$$\left. - zZe^2\int_{r_0}^\infty dr\, r^{-2}\{[F_1(\eta', k'r) + G_1(\eta', k'r)\tan\alpha][G_0(\eta, kr) + iF_0(\eta, kr)]\}\right\} + O(\Delta).$$
$$(5)$$

We separate the expression (5) into real and imaginary contributions and consider the latter first.

We may neglect the contribution of the term $F_0(\eta, kr)G_1(\eta', k'r)\tan\alpha$ to the integral since it is of order $O(\Delta)$. Thus, the imaginary part is an integral over two F_j functions. Therefore it contains those contributions to the bremsstrahlung that stem from the classical acceleration in the Coulomb field. It can be treated analytically as follows. We first extend the lower

limit of the integral to zero, which only introduces an error of the order $O(\Delta)$. The resulting integral may be expressed in terms of hypergeometric functions as [8,9]

$$\int_0^\infty \frac{dr}{r^2} F_1(\eta', k'r)F_0(\eta, kr)$$
$$= kk'\{k'|1 + i\eta'|M_0 - k|1 + i\eta|M_1\}, \quad (6)$$

where

$$M_j = \left(\frac{\xi}{\eta + \eta'}\right)^{i(\eta+\eta')}\frac{|\Gamma(j + 1 + i\eta')|\,|\Gamma(j + 1 + i\eta)|}{(k - k')^2(2j + 1)!} e^{-(\pi/2)\xi}\left(\frac{\eta'\eta}{\xi^2}\right)^j$$
$$\times {}_2F_1\left(j + 1 - i\eta, j + 1 - i\eta', 2j + 2; -\frac{\eta'\eta}{\xi^2}\right).$$
$$(7)$$

Here ${}_2F_1$ denotes the hypergeometric function, and we have defined [10]

$$\xi = \eta' - \eta. \quad (8)$$

In the limit of vanishing photon energy the imaginary part of the matrix element (5) may be computed directly, using [12]

$$\lim_{k'\to k} \text{Im}\langle\Phi_f|\partial_r V|\Phi_i\rangle = -\sqrt{\frac{mE_\alpha}{\pi\hbar}}\frac{\eta}{\sqrt{1 + \eta^2}}. \quad (9)$$

The real part of the matrix element (5) is a sum of two terms which, in contrast to the imaginary part, involve contributions from the irregular Coulomb wave functions G_j. Thus, it describes those contributions to the

bremsstrahlung that are associated with tunneling. In the limit of vanishing photon energy, this amplitude reduces to [12,13]

$$\lim_{k'\to k} \text{Re}\langle\Phi_f|\partial_r V|\Phi_i\rangle = \sqrt{\frac{mE_\alpha}{\pi\hbar}}\frac{1}{\sqrt{1 + \eta^2}}. \quad (10)$$

Notice that the dependence on the inner barrier parameters has disappeared. A comparison with the imaginary part (9) shows that the real part (10) is suppressed by a factor η. For nonzero photon energy we have to treat the real part of the matrix element (5) numerically. However, the numerical evaluation shows that the real part still is suppressed in comparison to the imaginary part. This

implies that only a smaller fraction of bremsstrahlung is emitted during tunneling. Note also that the contributions associated with classical acceleration and tunneling do not interfere since they differ in phase by i.

We will now make the connection to semiclassical and classical limits. For heavy nuclei, the Sommerfeld parameters η are large and the Coulomb wave functions F_j may be approximated by their WKB wave functions

$$F_j^{\text{WKB}}(\eta, kr) = [k^2/f(r)]^{1/4} \sin \phi, \qquad (11)$$

with

$$f(r) = k^2 - 2k\eta/r - j(j+1)/r^2 \quad \text{and} \qquad (12)$$

$$\phi = \frac{\pi}{4} + \int_{2\eta/k}^{r} dr' [f(r')]^{1/2}. \qquad (13)$$

In leading order in η, η' one finds [9]

$$\int_{2\eta}^{\infty} \frac{dr}{r^2} F_1^{\text{WKB}}(\eta', k'r) F_0^{\text{WKB}}(\eta, kr) \approx -\frac{kk'}{k+k'} \frac{\xi}{\eta} e^{-(\pi/2)\xi} \left[K_{i\xi}'(\xi\epsilon) + \frac{(\epsilon^2-1)^{1/2}}{\epsilon} K_{i\xi}(\xi\epsilon) \right], \qquad (14)$$

where $\epsilon = \frac{(\overline{\eta}^2 + 3/4)^{1/2}}{\overline{\eta}}$ and $\overline{\eta} = (\eta' + \eta)/2$. $K_\nu(z)$ denotes the modified Bessel function and K_ν' its derivative with respect to the argument.

A comparison of the semiclassically evaluated integral (14) with the quantum mechanical result (6) shows that they deviate from each other by less than 1% for photon energies E_γ up to 1 MeV. We recall that the semiclassical computation neglects any contributions from the wave functions at radii smaller than the classical turning point, i.e., any contribution from the tunneling. This clearly justifies the attribution of tunneling to the real part of the matrix element, Eq. (5), alone.

Next we consider the classical and the soft photon limit. The classical formula valid at all frequencies can be derived from [14]

$$\frac{dP}{dE_\gamma} = \frac{2\alpha Z_{\text{eff}}^2}{3\pi} \frac{|I(\omega)|^2}{E_\gamma}, \qquad (15)$$

with I the Fourier transform of the time-dependent acceleration,

$$I(\omega) = c^{-1} \int_0^{\infty} dt \frac{dv}{dt} \exp(i\omega t). \qquad (16)$$

This integral can be expressed in terms of the dimensionless parameter

$$\zeta = \eta \frac{\hbar\omega}{E_\alpha} \qquad (17)$$

as

$$I(\omega) = \sqrt{\frac{2E_\alpha}{mc^2}} \int_0^1 dz \exp\left[i\zeta \left(\frac{z}{1-z^2} + \text{artanh}\, z \right) \right]. \qquad (18)$$

In the limit of small photon energy we find

$$\frac{dP}{dE_\gamma} = \frac{4\alpha Z_{\text{eff}}^2}{3\pi} \frac{E_\alpha}{mc^2} E_\gamma^{-1} \equiv C, \qquad (19)$$

defining C. Because this depends only on the asymptotic motion of the particles, the quantum result must coincide. Inserting the results (9) and (10) into Eq. (2) indeed yields the classical result, Eq. (19).

More interesting is to examine the next-to-leading E_γ dependence and compare the quantum and classical behavior. It turns out that the classical parameter ζ in Eq. (17) is essentially the same as the quantum small parameter ξ defined in Eq. (8) $[\zeta = 2\xi + O(\xi^2/\eta)]$. This parameter may also be identified with the product of the photon frequency and the modulus of the imaginary barrier tunneling time. The ζ dependence of the classical and quantum calculations are compared in Fig. 1. The solid line shows the classical prediction (15). The dashed and dotted lines show the quantum result with and without tunneling contributions, respectively. We see that the tunneling contributions remain small even at a finite photon energy.

One might have expected that the classical curve would be tangent to the quantum at $\omega = 0$: in scattering bremsstrahlung is determined by on-shell amplitudes to next-to-leading order [15]. We find that the two curves are indeed very close in the neighborhood $\omega = 0$, but the slopes are not identical. For large photon energies the classical result overestimates the photon emission rate considerably since the classical formula (15) neglects any energy loss of the escaping α particle. This point has
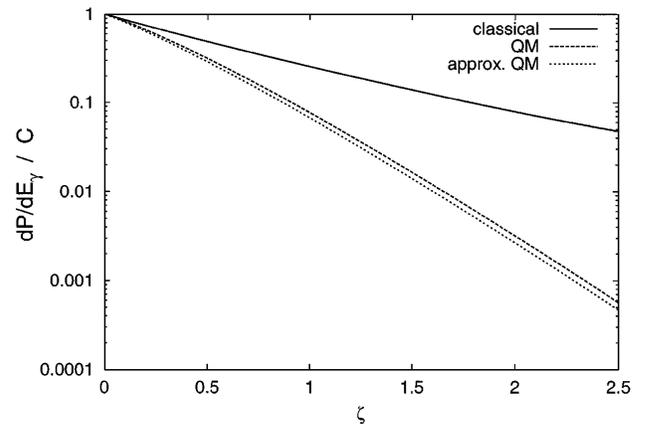
FIG. 1. Comparison of classical and quantum mechanical photon emission probability in α decay of ^{210}Po. Curves show the probabilities normalized to the low-energy expression C defined in Eq. (19), as a function of the scaled photon frequency ζ defined in Eq. (17). The classical probability, Eq. (15), is shown as the solid line. The quantum probabilities (nearly exponentially falling lines) are shown for the full quantum mechanical treatment (dashed line) and for the approximation that neglects contribution from tunneling (dotted line). $\zeta = 1$ corresponds to $E_\gamma \approx 0.24$ MeV.
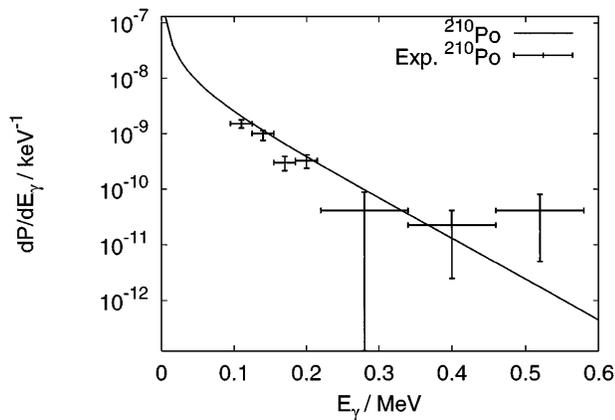
FIG. 2. Photon emission probability comparing the quantum calculation with experiment. Logarithmic plot of $\frac{dP}{dE_\gamma}$ as a function of photon energy for ^{210}Po (full line). The experimental data for ^{210}Po (data points with error bars) are taken from Ref. [4].

been discussed in the framework of photon emission in spontaneous fission by Luke *et al.* [2], and earlier in the framework of Coulomb excitation by Alder *et al.* [9].

The quantum mechanical results for ^{214}Po and ^{226}Ra are practically identical to those for ^{210}Po when plotted as in Fig. 1, normalized to the $\omega = 0$ rate (19) and plotted as a function of ζ. Since ζ is inversely proportional to the decay energy E_α, the rates are higher for higher decay energies. Thus for ^{214}Po decay, with an α-decay energy of 7.7 MeV, the predicted rate for $E_\gamma = 0.6$ MeV is 65 times higher than for ^{210}Po.

Finally, we compare the results obtained in this work with experiment. In the case of ^{210}Po, our result displayed in Fig. 2 is consistent with the experimental result obtained by Kasagi *et al.* [4] suggesting that no interference resulting from photon emission during tunneling is needed for an explanation of the experiment. In the case of ^{214}Po and ^{226}Ra, D'Arrigo *et al.* [3] reported photon emission rates that are larger than expected from the classical formula (15). Thus, their results are also more than 1 order of magnitude larger than our mechanically computed quantum rate. We cannot trace the origin of this difference.

In summary, we have used Fermi's golden rule to compute the emission of bremsstrahlung in α decay of heavy nuclei. The dominant contribution to the photon emission rate stems from classical acceleration and is given in closed form. Only a smaller fraction of bremsstrahlung is emitted during tunneling. This finding is consistent with experimental data on ^{210}Po.

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