Lattice calculations for TMDs and DPDs


- TMDs on the lattice
- The relevance of DPDs
- DPDs on the lattice “Two-current correlations in the pion on the lattice” arXiv:1807.03073
- Conclusion
A few basic facts relevant for this talk
QCD is contained in the generating functional:

\[ Z[J^a_\mu, \bar{\eta}^i, \eta^i] = \int \mathcal{D}[A^{a\mu}, \bar{\psi}^i, \psi^i] \exp \left( i \int d^4x \left[ \mathcal{L}_{\text{QCD}} - J^a_\mu A^{a\mu} - \bar{\psi}^i \eta^i - \bar{\eta}^i \psi^i \right] \right) \]

A numerical integration is made possible by analytic continuation to imaginary time:

\[ t \leftrightarrow -i\tau \]

\[ S = \int d^4x (T - V) \leftrightarrow i \int d^4x_E (T + V) = iS_E \]

\[ e^{iS} \leftrightarrow e^{-S_E} \]
Discretized space time ⇒ e.g. the Wilson action

\[ U(l_1) = \exp \left( -igA^b(l_1) \frac{\lambda^b}{2} a \right) \]

\[ W_\square = \text{Tr}\{U(l_1)U(l_2)U(l_3)U(l_4)\} \]

\[ \sum_\square \frac{2}{g^2}(3 - \text{Re } W_\square) = \frac{1}{4} \int d^4x \left( F^a_{\mu\nu}F^a_{\mu\nu} + O(a^2) \right) \]
The $U(\ell)$ are the building blocks of Lattice QCD ⇒ Investigating gauge links should be simple
$$\int D[\psi, \bar{\psi}]... \Rightarrow \text{Det}[D]$$

However, non-locality in time cannot be treated numerically ⇒ Mellin moments; quasi PDFs

The continuum limit $a \rightarrow 0$ is the numerical challenge
large topological autocorrelation times for $a < 0.05$ fm. Usually $0.05$ fm $< a < 0.1$ fm

For distances of a few lattice spacings discretization errors are typically large ⇒ $1/[O(3)a]$ must be larger than any physically relevant momentum.

Simulations with open, not periodic boundary conditions M. Lüscher and S. Schaefer ⇒ CLS, a collaboration od collaborations
Hadronic 2- and 3- Point functions

One needs combinations of field operators which have the wanted quantum numbers, e.g. for the nucleon \((C = i\gamma^2\gamma^4 = C^{-1})\):

\[
\hat{B}_\alpha(t, \vec{p}) = \sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \epsilon_{ijk} \hat{u}_\alpha^i(x) \hat{u}_\beta^j(x)(C^{-1}\gamma_5)_{\beta\gamma} \hat{d}_\gamma^k(x)
\]

\[
\langle 0 | T \left\{ \hat{B}(y_4)\hat{A}(x_4) \right\} | 0 \rangle = e^{-(T-y_4+x_4)E_B} \langle B | \hat{B}(0) | 0 \rangle \langle 0 | \hat{A}(0) | B \rangle + e^{-(y_4-x_4)E_A} \langle 0 | \hat{B}(0) | A \rangle \langle A | \hat{A}(0) | 0 \rangle
\]

\(\hat{B}\) generates the antiparticle of \(\hat{A}\). One has (anti)periodic boundary conditions.
To get the hadron masses one simply has to determine the slopes.

\[ e^{-(y_4-x_4)M_N} \langle 0| \hat{N}^\dagger(0)|N\rangle \langle N|\hat{N}(0)|0\rangle \]

\[ |B\rangle \sim c_0|N\rangle + c_1|N'\rangle + c_2|N_{\pi}\rangle + \ldots \]

\[ \Rightarrow c_0 e^{-E_N t}|N\rangle + c_1 e^{-E_{N'} t}|N'\rangle + c_2 e^{-E_{N_{\pi}} t}|N_{\pi}\rangle + \ldots \]

Note: A quark propagator is the inverse of the Dirac operator on the lattice, which is just a large matrix.

\[ \langle B_\alpha(t, \vec{p})\bar{B}_\beta(0, \vec{p}) \rangle \]

\[ = \sum_{x_{4}=t} \sum_{y_{4}=0} \epsilon^{i\vec{p} \cdot (\vec{x} - \vec{y})} \epsilon_{ijk} \epsilon_{i'j'k'} (C^{-1}\gamma_5)_{\alpha'\alpha''} (\gamma_5 C)_{\beta'\beta''} \]

\[ \bigg( G_{\alpha''\beta'}^{ki'}(x, y) \big( G_{\alpha'\beta''}^{ij'}(x, y) G_{\alpha\beta}^{ik'}(x, y) - G_{\alpha'\beta''}^{ij'}(x, y) G_{\alpha'\beta}(x, y) \big) \bigg) \bigg|_g \]
A nucleon 2-point function
Once the propagation in imaginary time has projected the original source onto the physical wave function on can calculate physical correlators from

\[ \tilde{\Gamma}_{\alpha\beta} \langle B_\beta(t, \vec{p}) \bar{B}_\alpha(0, \vec{p}) \rangle \]

\[ \frac{\Gamma_{\alpha\beta} \langle B_\beta(t, \vec{p}) \bar{B}_\alpha(0, \vec{p}) \rangle}{\Gamma_{\alpha\beta} \langle B_\beta(t, \vec{p}) \bar{B}_\alpha(0, \vec{p}) \rangle} \]
TMDs are related to correlators of the type

$$\tilde{\phi}_{\text{unsubtr.}}^{[\Gamma]} (b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma U[0, \eta v, \eta v+b, b] q(b) | P, S \rangle$$

We simulate for spatial, not light-like separations, but the limit $\hat{\zeta} \to \infty$ of

$$\hat{\zeta} := \frac{v \cdot P}{\sqrt{v^2} \sqrt{P^2}}$$

reproduces the light-cone behavior.
We used RBC/UKQCD (domain wall) and W&M (Clover) ensembles, $N_f = 2 + 1$

<table>
<thead>
<tr>
<th>ID</th>
<th>Clover</th>
<th>DWF</th>
</tr>
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<tbody>
<tr>
<td>Fermion Type</td>
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<td>Domain-wall</td>
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<td>$32^3 \times 64$</td>
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<td>$m_\pi$(MeV)</td>
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<td># confs.</td>
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<td>533</td>
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<tr>
<td># meas.</td>
<td>23208</td>
<td>4264</td>
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</table>

only connected diagrams, i.e. $u - d$
\[ \tilde{\Phi}^{[\Gamma]}_{\text{subtr.}}(b, P, S, \ldots) = \tilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \cdot S \cdot Z_{\text{TMD}} \cdot Z_2 \]

\[ \Phi^{[\Gamma]}(x, k_T, P, S, \ldots) = \int \frac{d^2b_T}{(2\pi)^2} \int \frac{d(b \cdot P)}{2\pi P^+} e^{ix(b \cdot P) - ib_T \cdot k_T} \tilde{\Phi}^{[\Gamma]}_{\text{subtr.}} \bigg|_{b^+ = 0} \]

\[ \Phi^{[\gamma^+]} = f_1 - \frac{\epsilon_{ij} k_i S_j}{m_N} \tilde{f}_{1T} \]

\[ \Phi^{[\gamma^+ \gamma^5]} = \Lambda g_1 + \frac{k_T \cdot S_T}{m_N} g_{1T} \]

\[ \Phi^{[i\sigma^i \gamma^5]} = S_i h_1 + \frac{(2k_i k_j - k_T^2 \delta_{ij}) S_j}{2m_N^2} h_{1T} + \frac{\Lambda k_i}{m_N} h_{1L} + \frac{\epsilon_{ij} k_j}{m_N} h_{1L} \]

\[ \tilde{f}^{[m]}(n)(b_T^2, \ldots) = n! \left(-\frac{2}{m_N^2} \partial_{b_T^2}\right)^n \int_{-1}^1 dx x^{m-1} \int d^2k_T e^{ib_T \cdot k_T} f(x, k_T^2) \]

\[ \langle \tilde{k}_y \rangle_{TU}(b_T^2; \ldots) = m_N \frac{\tilde{f}_{1T}^{[1]}(b_T^2; \ldots)}{\tilde{f}_{1}^{[1]}(0)(b_T^2; \ldots)} \]
the generalized tensor charge

$$g_T^{u-d} = \int dx \, d^2 k_T \, h_1(x, k_T^2) = \tilde{h}_1^{[1](0)}(b_T^2 = 0)$$

limits:

$$\eta |v| \rightarrow \infty$$
$$b_T \gg a$$
$$\zeta \rightarrow \infty$$
Sivers shift
Boer-Mulders shift

Boer–Mulders Shift

\[
\frac{m_N h_1^{[4]}(1)}{f_1}\quad \text{(GeV)}
\]

\( \hat{\zeta} = 0.32, \quad |b_\gamma| = 0.34 \text{ fm}, \quad \text{Clover} \)

DY \quad \text{SIDIS}

\( \hat{\zeta} = 0.41, \quad |b_\gamma| = 0.25 \text{ fm}, \quad \text{DWF} \)

DY \quad \text{SIDIS}

Boer–Mulders Shift

\[
\frac{m_N h_1^{[4]}(1)}{f_1}\quad \text{(GeV)}
\]

\( \hat{\zeta} = 0.32, \quad |b_\gamma| = 0.46 \text{ fm}, \quad \text{Clover} \)

DY \quad \text{SIDIS}

\( \hat{\zeta} = 0.41, \quad |b_\gamma| = 0.34 \text{ fm}, \quad \text{DWF} \)

DY \quad \text{SIDIS}
Worm-gear shift, for $g_{1T}$
note

- Discrepancy between Clover and DWF for large $\eta$ and small $b_T$
- Very funny behaviour for $\eta = 0$, i.e. for a straight gauge link. Not relevant for TMDs but perhaps a hint for an explanation of the first observation
transversity ratio
Sivers shift

\[ \hat{\zeta} \approx 0.3 \]

Sivers Shift (SIDIS)

\[ \frac{m_N f_{1T}}{f_1} \]

\[ m_N f_{1T} \]

\[ \hat{\zeta} = 0.32 \]
\[ \hat{\zeta} = 0.22 \]
\[ \hat{\zeta} = 0 \]
Boer–Mulders shift
Worm-gear shift, for $g_{1T}$

M. Constantinou, H. Panagopoulos arXiv:1705.11193 operator mixing
Operator-mixing under renormalization for the worm-gear shift and Clover-Wilson fermions
There exists a continuum analysis of I. Scimemi and A. Vladimirov arXiv:1804.08148

$$\frac{g^{(0)}_{1T}(\vec{b})}{f^{(0)}_{1}(\vec{b})} \sim 0.13 + \frac{\Delta T^{(2,1)}}{2f^{(0)}_{1}} + O(\alpha_s) + O(\vec{b}^2) \sim 0.13$$

if $\Delta T^{(2,1)}$ as well as all higher order and large distance effects are set equal to zero, to be compared to the lattice value of roughly 0.2.
Transversity $h_1\(^{(1)}(0)\) / \hat{f}_1(0)$

$
\hat{\zeta} \approx 0.3
$

$\hat{\zeta} = 0$
$\hat{\zeta} = 0.22$
$\hat{\zeta} = 0.32$

**transversity ratio**
Sivers Shift (SIDIS)

\[ |b_T| = 0.34 \text{ fm} \]

\[ |b_T| = 0.68 \text{ fm} \]

Sivers shift

\[ m_N \frac{\tilde{f}_1^{[1]}(1)}{\tilde{f}_1^{[1]}(0)} \]
Boer–Mulders Shift (SIDIS)

\[ |b_T| = 0.34 \text{ fm} \]

\[ |b_T| = 0.68 \text{ fm} \]

Boer–Mulders shift
Worm-gear shift, for $g_{1T}$
transversity ratio
Gen. Sivers Shift (SIDIS, u-d; GeV) exp. estimate, $|b_T| \approx 0.35$ fm
DWF-on-AsqTad; 0.12 fm, 518 MeV
DWF; 0.084 fm, 297 MeV
Clover; 0.114 fm, 317 MeV

Comparison with experiment

$|b_T| \approx 0.68$ fm
Improvement strategy (for a decade ??)

- physical masses, finer lattices, more statistics etc. straightforward but numerically extremely demanding
- Simulating for larger $\zeta$, i.e. larger momentum

**momentum dependent smearing:** B. Musch et al., 1602.05525
Something new: TMD evolution

Various groups have produced fits to existing data. Some results from


\[
F_{q \leftarrow N}(x, \vec{b}; \zeta, \mu) = \frac{Z_q(\zeta, \mu)R_q(\zeta, \mu)}{2} \times \sum_{\chi} \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \langle N | \left\{ T[\bar{q}_i \tilde{W}_n^\dagger q_j]_a \left( \frac{\xi}{2} \right) | X \rangle \times \gamma_{ij}^+ \langle X | T[\tilde{W}_n^\dagger q_j]_a \left( -\frac{\xi}{2} \right) \rangle | N \rangle, \\
F(x, \vec{b}; \zeta_f, \mu_f) = \mathcal{R}(\vec{b}; \zeta_f, \mu_f, \zeta_i, \mu_i)F(x, \vec{b}; \zeta_i, \mu_i) \\
\mathcal{R}(\vec{b}; \zeta_f, \mu_f, \zeta_i, \mu_i) = \exp \left\{ \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma \left( \alpha_s(\mu), \ln \frac{\zeta_f}{\mu^2} \right) \right\} \left( \frac{\zeta_f}{\zeta_i} \right)^{-\mathcal{D}(\mu_i, \vec{b})}
\]
TMDs themselves as well as their evolution contain unsuppressed non-perturbative parts, which have to be parameterized.

\[
F_{q \leftarrow h}(x, \vec{b}; \mu, \zeta) = \int_x^1 \frac{dz}{z} \sum_f C_{q \leftarrow f}(z, \vec{b}; \mu, \zeta) f_{f \leftarrow h}\left(\frac{x}{z}, \mu\right) f_{NP}(z, \vec{b})
\]

\[
f_{NP}(\vec{b}) = e^{-\lambda_1 b}(1 + \lambda_2 b^2)
\]

\[
D^f(\mu, \vec{b}) = \int_{\mu_0}^\mu \frac{d\mu'}{\mu'} \Gamma_f + D_{\text{pert}}^f(\mu_0, \vec{b}) + g_K \vec{b}^2
\]
experimental (fat) and theoretical (thin; from just varying $\mu$ and the coefficient function in reasonable bounds) uncertainties for TMDs from DY
What next? study the non-perturbative part of evolution This requires many different $a = 1/\mu \Rightarrow \text{CLS}$
continuum QCD predicts that all TMDs have the same scaling, which, therefore, can be obtained from all the ratios

\[
\frac{f[m,n](\zeta)}{f[m,n](\zeta')} = \frac{f[m',n](\zeta)}{f[m',n](\zeta')} = \left(\frac{\zeta}{\zeta'}\right)^\left(-D_{\text{pert}}+D_{\text{NP}}(\mu,\vec{b}_T)\right)
\]

\[D(\mu,\vec{b}_T)_{\text{pert}} = C_f \sum_{n=1}^{\infty} a_s^n \sum_{k=0}^{n} L_{\mu}^k d^{(n,k)}\]

\[L_{\mu} := \ln \frac{\mu^2 b_T^2}{4e^{-2\gamma_E}}\]

The \(d^{(n,k)}\) are known. This tests also the assumption that on the lattice one has multiplicative renormalisation factors which cancel in ratios.
Calculate correlators non-local in space on the lattice and relate them to PDFs, DAs etc.

\[ q(x, \mu^2) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^-P^+} \langle P|\bar{\psi}(\xi^-)\gamma^+ \exp\left(-ig\int_0^{\xi^-} d\eta^- A^+_{\eta^-}\right)|\psi(0)\rangle \]

\[ q(x, \mu^2, P^z) = \int \frac{dz}{4\pi} e^{izk^z} e^{-ix\xi^-} \langle P|\bar{\psi}(z)\gamma^z \exp\left(-ig\int_0^{z} dz' A^z(z')\right)|\psi(0)\rangle \]

\[ + \mathcal{O}\left( (\Lambda_{QCD}/P^z)^2, (M/P^z)^2 \right) \]

\[ q(x, \mu^2, P^z) = \int_x^1 \frac{dy}{y} Z\left(\frac{x}{y}, \frac{\mu}{P^z}\right) q(y, \mu^2) \]

\[ Z(x, \mu/P^z) = \delta(x - 1) + \frac{\alpha_s}{2\pi} Z^{(1)}(x, \mu/P^z) + \ldots \]

Note: One has to fulfill \( \mu = 1/x \geq 1 \text{ GeV and } P^z/\mu = P^z x \sim 1 \).
The idea is great, but how large are the systematic errors? The LP$^3$ collaboration and others explore quark quasi-distribution functions systematically. This works nice for the nucleon quark PDFs arXiv:1807.06566.
is more difficult for DAs arXiv:1712.10025

Left: Comparison of $\phi_\pi$ to previous determinations in literature.
Right: Converted of $\phi_\pi$ to the scheme of Braun and Mueller
is not very successful for the pion PDFs arXiv:1804.01483

There was/is (?) a big debate about the pion quark PDF
ASV is Aicher, AS, Vogelsang arXiv:1009.2481
My very personal opinion: While it is nice to explore ever more quantities, the real challenge is to understand the systematic errors of the quasi PDF/DA/TMD approach. Most puzzleing: Why does the method work much better for the nucleon than for the pion?

In addition there are still conceptual questions to be solved, e.g. renormalization for quasi gluon PDFs


Independently the same was done with different techniques but similar results by Z. Y. Li, Y. Q. Ma and J. W. Qiu, arXiv:1809.01836
What are Double Parton Distributions (DPDs)?
Why are they related to TMDs?
LHC: DPIs (more generally MPIs) look unproblematic at first sight

but they are not !!!

- collinear physics ⇒ light-cone gauge ⇒ gauge links can be neglected. **Here they can not; connection to TMDs**
- high multiplicity = enhanced MPI contributions
mean charged particle $p_T$ as function of $N_{ch}$.

Events with large $N_{ch}$ have an even higher MPI contribution. MPIs produce many particles with $p_T$ of $O(1\ \text{GeV})$. 

$Z \rightarrow \mu^+\mu^-$, 7 TeV, Transverse region, dressed level

$Z \rightarrow \mu^+\mu^-$, 7 TeV, Away region, dressed level
The connection: A. Vladimirov arXiv:1608.04920

Abstract: We show at NNLO that the soft factors for double parton scattering (DPS) for both integrated and unintegrated kinematics, can be presented entirely in the terms of the soft factor for single Drell-Yan process, i.e. the transverse momentum dependent (TMD) soft factor ...
Factorizing the soft factor
The fourth order cumulant
\[ v_n\{4\} = \left( \left\langle \left\langle e^{in(\phi_1+\phi_2-\phi_3-\phi_4)} \right\rangle \right\rangle_c \right)^{1/4} \]
for p+p.

“These findings indicate that the no-interaction baseline including QCD interference effects can make a sizeable if not dominant contribution to the measured \( v_n \) coefficients in pp collisions.”
A large DESY (M. Diehl)-Regensburg project: Moments of DPD’s on the lattice ch. 4.2 in arXiv: 1111.0910

\[ M_{a_1,a_2}^{n_1,n_2}(y^2) = \int_0^1 dx_1 x_1^{n_1-1} \int_0^1 dx_2 x_2^{n_2-1} \left[ 1F_{a_1,a_2}(x_1,x_2,y) + (-1)^{n_1} \sigma_{a_1} 1F_{\bar{a}_1,a_2}(x_1,x_2,y) \right] 
+ (-1)^{n_2} \sigma_{a_2} 1F_{a_1,\bar{a}_2}(x_1,x_2,y) + (-1)^{n_1+n_2} \sigma_{a_1} \sigma_{a_2} 1F_{\bar{a}_1,\bar{a}_2}(x_1,x_2,y) \right] 
= \frac{1}{2} (p^+)^{1-n_1-n_2} \int dy^- \langle p | O_{a_1}^{+\cdots+}(0) O_{a_2}^{+\cdots+}(y) | p \rangle_{y^+ = 0} \]

\[ O(q^{\mu_1\cdots\mu_n}(y) = \frac{T}{(\mu_1\cdots\mu_n)(\mu_1\cdots\mu_n)} \bar{q}(y) \gamma^{\mu_1} i\not\!D^{\mu_2}(y) \cdots i\not\!D^{\mu_n}(y) q(y) \]

\[ O_{\Delta q}^{\mu_1\cdots\mu_n}(y) = \frac{T}{(\mu_1\cdots\mu_n)(\mu_1\cdots\mu_n)} \bar{q}(y) \gamma^{\mu_1} \gamma_5 i\not\!D^{\mu_2}(y) \cdots i\not\!D^{\mu_n}(y) q(y) \]

\[ O_{\delta q}^{\lambda\mu_1\cdots\mu_n}(y) = \frac{T}{(\lambda\mu_1\cdots\mu_n)(\lambda\mu_1)(\mu_1\cdots\mu_n)} \bar{q}(y) i\sigma^{\lambda\mu_1} \gamma_5 i\not\!D^{\mu_2}(y) \cdots i\not\!D^{\mu_n}(y) q(y) \]

\[ i\not\!D^{\mu}(y) = \frac{1}{2} (\partial^{\mu} - \partial^{\mu}) + igA^{\mu}(y) \]

arXiv:1807.03073 \( n = 1 \), charge-charge correlations in the pion; hopefully the first of many papers
Correlations of interest are linear combinations of quantities calculated separately on the lattice ⇒ High numerical accuracy is needed. Therefore, we started with the pions.

\[
\begin{align*}
\langle \pi^+ | O_{i}^{uu}(y) O_{j}^{dd}(0) | \pi^+ \rangle &= C_1 + [2S_1 + D] \\
\langle \pi^+ | O_{i}^{uu}(y) O_{j}^{uu}(0) | \pi^+ \rangle &= [2C_2 + S_2] + [2S_1 + D] \\
\langle \pi^0 | O_{i}^{uu}(y) O_{j}^{dd}(0) | \pi^0 \rangle &= [2S_1 + D] - A \\
\langle \pi^0 | O_{i}^{uu}(y) O_{j}^{uu}(0) | \pi^0 \rangle &= C_1 + [2S_1 + D] + [2C_2 + S_2] + A \\
\langle \pi^0 | O_{i}^{du}(y) O_{j}^{du}(0) | \pi^0 \rangle &= -C_1 + [2C_2 + S_2] \\
\langle \pi^- | O_{i}^{du}(y) O_{j}^{ud}(0) | \pi^+ \rangle &= 2C_1 + 2A \\
\langle \pi^+ | O_{i}^{du}(y) O_{j}^{du}(0) | \pi^+ \rangle &= 2C_2(y) + S_2 + A^{ij}(y) \\
\sqrt{2} \langle \pi^0 | O_{i}^{du}(y) O_{j}^{uu}(0) | \pi^+ \rangle &= C_1 + [C_2^{ij}(y) - C_2^{ij}(-y)] + A^{ij}(y)
\end{align*}
\]
\( \mathcal{C}^{ij}(y) = \)

\[ O_i(y) \]

\[ O_j(0) \]

\( \mathcal{C}^{ij}_2(y) = \)

\[ O_j(0) \]

\[ O_i(y) \]

\vphantom{\times}\mathcal{C} \times \mathcal{C}^{ij}_2(y) = \eta^{ij}_C \times \]

\[ O_j(0) \]

\[ O_i(y) \]

\( \mathcal{A}^{ij}(y) = \)

\[ O_i(y) \]

\[ O_j(0) \]
\[ S_{1}^{ij}(y) = O_{i}(y) \times O_{j}(0) = \eta_{ij} \times \]

\[ S_{2}^{ij}(y) = O_{i}(y) \times O_{j}(0) \]

\[ D_{ij}(y) = O_{i}(y) \times O_{j}(0) \]
On the lattice this is calculated like this
We have started meanwhile also simulations for the nucleon, which is the case relevant for LHC.
All stays the same except for $C_1$.

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$\beta$</th>
<th>$a$ [fm]</th>
<th>$\kappa$</th>
<th>$L^3 \times T$</th>
<th>$m_\pi$ [MeV]</th>
<th>$Lm_\pi$</th>
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<th>$N_{\text{used}}$</th>
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<tr>
<td>V</td>
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<td>288.8(11)</td>
<td>4.19</td>
<td>2025</td>
<td>984</td>
<td>400</td>
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The used $N_f = 2$ ensembles.
Renormalization:

$$R_{ij}^{\text{MS}} = \tilde{Z}_i \tilde{Z}_j R_{ij}^{\text{lat}} \quad \text{with} \quad \tilde{Z}_i = Z_i^{\text{MS}} (1 + am_q b_i)$$

<table>
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<th></th>
<th>$S$</th>
<th>$P$</th>
<th>$V$</th>
<th>$A$</th>
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<td>0.6153(25)</td>
<td>0.476(13)</td>
<td>0.7356(48)</td>
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<tr>
<td>$b_{\text{pert}}^{\text{pert}}$</td>
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<td>1.2747</td>
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<td>1.586(32)</td>
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<tr>
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<td>1.586</td>
<td>1.586</td>
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</tbody>
</table>

Renormalisation factors $Z_i^{\text{MS}}$ from Bali et al. arXiv:1412.7336 in the $\overline{\text{MS}}$ scheme at $\mu = 2\text{GeV}$. For $P$, $A$ and $T$ we rescale the perturbative value according to $V$. The $S$ case shows that this procedure contributes a significant systematic error.
very few results

\[ \langle V^0 V^0 \rangle, p^2 = 0, L = 40 \]

\[ \langle A^0 A^0 \rangle, p^2 = 0, L = 40 \]

\[ \langle S S \rangle, p^2 = 0, L = 40 \]

\[ \langle P P \rangle, p^2 = 0, L = 40 \]
quark mass dependence

\[ \langle V^0V^0 \rangle, |C_1|, p^2 = 0, L = 40 \] (log scale)

\[ \langle S\bar{S} \rangle, |C_1|, p^2 = 0, L = 40 \] (log scale)

\[ \langle P\bar{P} \rangle, |C_1|, p^2 = 0, L = 40 \] (log scale)
The strength of correlation effects

\[ M_{ii}(\vec{q}^2) = \int d^3 y \ e^{i\vec{y} \cdot \vec{q}} \left\langle \pi^+(p) \right| \mathcal{O}^{uu}_i(y) \mathcal{O}^{dd}_i(0) \left| \pi^+(p) \right\rangle \]

\[ \overset{?}{=} \int d^3 y \ e^{i\vec{y} \cdot \vec{q}} \int \frac{d^3 p'}{(2\pi)^3 2p'0} \left\langle \pi^+(p) \right| \mathcal{O}^{uu}_i(y) \left| \pi^+(p') \right\rangle \times \left\langle \pi^+(p') \right| \mathcal{O}^{dd}_i(0) \left| \pi^+(p) \right\rangle \]

\[ \overset{?}{=} \frac{\eta^i_C}{2E_q} \left| \left\langle \pi^+(E_q, -\vec{q}) \right| \mathcal{O}^{uu}_i(0) \left| \pi^+(p) \right\rangle \right|^2 \]

\[ -M_{V^0V^0}(\vec{q}^2) \overset{?}{=} \frac{(m_\pi + E_q)^2}{2E_q} \left[ F_V(2m_\pi E_q - 2m_\pi^2) \right]^2 \]

\[ M_{SS}(\vec{q}^2) \overset{?}{=} \frac{1}{2E_q} \left[ F_S(2m_\pi E_q - 2m_\pi^2) \right]^2 \]
The Fourier transformation of both sides of the factorization test equations.

⇒ The correlations are sizeable and must be treated exactly
⇒ Understanding transverse correlation in hadron structure could become crucial for LHC; work for decades
Conclusions

- Lattice QCD does not provide magic bullets. Progress is steady but very tedious.

- Calculating double moments of TMDs is in good shape but needs improvement (physical masses, continuum limit, larger $\zeta$, more statistics) and will never give more than a few Mellin moment.

- quasi-TMDs are a long term possibility to go beyond moments IF all systematic and conceptual problems can be controlled.

- MPIs are important for LHC. It is claimed that they can be perfectly described by event generators. I would like to see Lattice QCD checks. This is again very tedious and long term, but first results suggest significant quark correlations in the pion. Is there a difference between pions and nucleons?