Computing $\beta\beta$ Nuclear Matrix Elements

J. Engel

June 13, 2017
Light-$\nu$ Exchange Tells Us Neutrino Mass

$$[T_{1/2}^{0\nu}]^{-1} = \sum_{\text{spins}} \int |Z_{0\nu}|^2 \delta(E_{e1} + E_{e2} - Q) \frac{d^3p_1}{2\pi^3} \frac{d^3p_2}{2\pi^3}$$

Amplitude $Z_{0\nu}$ contains lepton part

$$\sum_k \bar{e}(x) \gamma_\mu (1 - \gamma_5) U_{ek} \nu_k(x) \overline{\nu}_k^c(y) \gamma_\nu (1 + \gamma_5) U_{ek} e^c(y),$$

where $\nu$'s are Majorana mass eigenstates.

Contraction gives neutrino propagator:

$$\sum_k \bar{e}(x) \gamma_\mu (1 - \gamma_5) \frac{q^\rho \gamma_\rho + m_k}{q^2 - m_k^2} \gamma_\nu (1 + \gamma_5) e^c(y) U_{ek}^2,$$

The $q^\rho \gamma_\rho$ part vanishes in trace, leaving a factor

$$m_{\text{eff}} \equiv \sum_k m_k U_{ek}^2.$$
Nuclear Part For Light-$\nu$ Exchange

\[ M_{O\nu} = M_{O\nu}^{GT} - \frac{g_V^2}{g_A^2} M_{O\nu}^F + \ldots \]

with

\[ M_{O\nu}^{GT} = \langle F | \sum_{i,j} H(r_{ij}) \sigma_i \cdot \sigma_j \tau_i^+ \tau_j^+ | I \rangle + \ldots \]

\[ M_{O\nu}^F = \langle F | \sum_{i,j} H(r_{ij}) \tau_i^+ \tau_j^+ | I \rangle + \ldots \]

\[ H(r) \approx \frac{2R}{\pi r} \int_0^{\infty} dq \frac{\sin qr}{q + \bar{E} - (E_i + E_f)/2} \quad \text{roughly } \propto 1/r \]

Contribution to integral peaks at \( q \approx 100 \text{ MeV} \) inside nucleus.

Corrections are from “forbidden” terms, weak nucleon form factors, many-body currents …
If neutrinoless decay occurs then \( \nu \)'s are Majorana, no matter what:

but light neutrinos may not drive the decay:

- Exchange of heavy right-handed neutrino in left-right symmetric model.

Amplitude of “exotic” mechanism:

\[
\frac{Z_{0\nu}^{\text{heavy}}}{Z_{0\nu}^{\text{light}}} \approx \left( \frac{M_{W_L}}{M_{W_R}} \right)^4 \left( \frac{\langle q^2 \rangle}{m_{\text{eff}} m_N} \right)
\]

\[\langle q^2 \rangle \approx 10^4 \text{ MeV}^2\]

\( \approx 1 \) if \( m_N \approx 1 \text{ TeV} \) and \( m_{\text{eff}} \approx \sqrt{\Delta m^2_{\text{atm}}} \)

So exotic stuff can occur with roughly the same rate as light-\( \nu \) exchange.

Upcoming talks by W. Rodejohann, M. Ramsey-Musolf
These lead to two-nucleon operators with different space dependence from that of the “standard” operator.

Upcoming talks by A. Nicholson, M. Graesser, M. Savage
Nuclear Matrix Elements: The Situation at Present

Light-ν Echange

Significant spread. And all the models could be missing important physics.

Uncertainty hard to quantify.
Nuclear Structure: Contrasting the Approaches

Starting point is always mean field(s)

Energy-Density Functional Theory employs Generator-Coordinate Method (GCM), which mixes many such states with different collective properties. Other methods build on single independent-particle state.

QRPA: Large single-particle spaces in arbitrary single mean field; simple correlations and excitations within the space.

Shell Model: Small single-particle space in simple spherical mean field; arbitrarily complex correlations within the space.

IBM is somewhere in between, mapping matrix elements from up to two shells but truncating to collective pairs.

Can we improve and combine these methods? Can we avoid fitting parameters to data directly in heavy nuclei? That's not a bad thing, but makes it hard to estimate accuracy when calculating something different from anything ever measured.
Nuclear Structure: Contrasting the Approaches

Starting point is always mean field(s)

\[ \cdots + \begin{array}{c}
\text{protons} \\
\text{neutrons}
\end{array} + \begin{array}{c}
\text{protons} \\
\text{neutrons}
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“Energy-Density Functional Theory” employs Generator-Coordinate Method (GCM), which mixes many such states with different collective properties.

Upcoming talk by T. Rodriguez
Nuclear Structure: Contrasting the Approaches

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Other methods build on single independent-particle state.

protons  neutrons

QRPA  Shell
Model

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The Way Forward: Ab Initio Nuclear Structure?

Often starts with chiral effective field theory.

Nucleons, pions sufficient below chiral-symmetry breaking scale.

<table>
<thead>
<tr>
<th>Order</th>
<th>2N Force</th>
<th>3N Force</th>
<th>4N Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO</td>
<td>(Q/Λχ)^0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NLO</td>
<td>(Q/Λχ)^2</td>
<td></td>
<td></td>
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<tr>
<td>NNLO</td>
<td>(Q/Λχ)^3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N^3LO</td>
<td>(Q/Λχ)^4</td>
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Figure 1: Hierarchy of nuclear forces in ChPT. Solid lines represent nucleons and dashed lines pions. Small dots, large solid dots, solid squares, and solid diamonds denote vertices of index = 0, 1, 2, and 4, respectively. Further explanations are given in the text.

The reason why we talk of a hierarchy of nuclear forces is that two- and many-nucleon forces are created on an equal footing and emerge in increasing number as we go to higher and higher orders. At NNLO, the first set of nonvanishing three-nucleon forces (3NF) occur [70, 71], cf. column '3N Force' of Fig. 1. In fact, at the previous order, NLO, irreducible 3N graphs appear already, however, it has been shown by Weinberg [52] and others [70, 127, 128] that these diagrams all cancel. Since nonvanishing 3NF contributions happen first at order (Q/Λχ)^3, they are very weak as compared to 2NF which start at (Q/Λχ)^0.

More 2PE is produced at \( \xi = 4 \), next-to-next-to-next-to-leading order (N^3LO), of which we show only a few symbolic diagrams in Fig. 1. Two-loop 2PE graphs show up for the first time and so does three-pion exchange (3PE) which necessarily involves two loops. 3PE was found to be negligible at this order [57, 58].

Most importantly, 15 new contact terms \( \varpi \) arise and are represented by the four-nucleon-leg graph with a solid diamond. They include a quadratic spin-orbit term and contribute up to D-waves. Mainly due to the increased number of contact terms, a quantitative description of the two-nucleon interaction up to about 300 MeV lab. energy is possible, at N^3LO (for details, see below). Besides further 3NF, four-nucleon forces (4NF) start at this order. Since the leading 4NF come into existence one order higher than the leading 3NF, 4NF are weaker than 3NF. Thus, ChPT provides a straightforward explanation for the empirically known fact that 2NF \( \ll 3NF \ll 4NF \) ...
The Way Forward: Ab Initio Nuclear Structure?

Often starts with chiral effective field theory.

Nucleons, pions sufficient below chiral-symmetry breaking scale.

Comes with consistent weak current.

Upcoming talk by S. Pastore
Ab Initio Shell Model

Partition of Full Hilbert Space

\( P \)
\( \hat{P} \hat{P} \)
\( \hat{P} \hat{Q} \)

\( Q \)
\( \hat{Q} \hat{P} \)
\( \hat{Q} \hat{Q} \)

\( P = \) valence space
\( Q = \) the rest

Task: Find unitary transformation to make \( H \) block-diagonal in \( P \) and \( Q \), with \( H_{\text{eff}} \) in \( P \) reproducing \( d \) most important eigenvalues.

Shell model done here.
Ab Initio Shell Model

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For transition operator $\hat{M}$, must apply same transformation to get $\hat{M}_{\text{eff}}$.

Upcoming talks by W.C. Haxton, L. Coraggio, H. Hergert

Shell model done here.
Ab Initio Shell Model

Partition of Full Hilbert Space

\[ P \quad H_{\text{eff}} \quad Q \]

\[ P = \text{valence space} \]
\[ Q = \text{the rest} \]

Task: Find unitary transformation to make \( H \) block-diagonal in \( P \) and \( Q \), with \( H_{\text{eff}} \) in \( P \) reproducing \( d \) most important eigenvalues.

For transition operator \( \hat{M} \), must apply same transformation to get \( \hat{M}_{\text{eff}} \).

As difficult as solving full problem. But idea is that N-body effective operators may not be important for \( N > 2 \) or 3.

Upcoming talks by W.C. Haxton, L. Coraggio, H. Hergert
One way to determine the transformation

Flow equation for effective Hamiltonian. Asymptotically decouples shell-model space.

\[
\frac{d}{ds} H(s) = [\eta(s), H(s)] , \quad \eta(s) = [H_d(s), H_{od}(s)] , \quad H(\infty) = H_{\text{eff}}
\]

Trick is to keep all 1- and 2-body terms in \( H \) at each step after normal ordering.

If shell-model space contains just a single state, approach yields ground-state energy. If it is a typical valence space, result is effective interaction and operators.
Ab Initio Calculations of Spectra

Neutron-rich oxygen isotopes

Deformed nuclei
Complementary Way Forward: Improved GCM?

Construct set of mean fields by constraining coordinate(s), e.g. quadrupole moment \( \langle Q_0 \rangle \). Then diagonalize (usually non-ab-initio) \( H \) in space of symmetry-restored quasiparticle vacua with different \( \langle Q_0 \rangle \).

\[ \beta_2 = \text{deformation} \]

Robledo et al.: Minima at \( \beta_2 \approx \pm 0.15 \)

Collective wave functions

Rodriguez and Martinez-Pinedo: Wave functions peaked at \( \beta_2 \approx \pm 0.2 \)

Can be improved by including crucial neutron-proton pairing amplitude as collective coordinate...
GCM Example: Proton-Neutron (pn) Pairing

Can build possibility of pn correlations into mean field. They are frozen out in mean-field minimum, but included in GCM.

Proton-neutron pairing significantly reduces matrix element.

Upcoming talk by N. Hinohara
GCM in Shell-Model Spaces

FIG. 2: Calculated low-lying excitation spectra of $^{76}\text{Ge}$ and $^{76}\text{Se}$ given by $pfsdg^{-2}$ interaction, compared with experimental data [5].

FIG. 3: The calculated occupancies of valence neutron and proton orbits for $^{76}\text{Ge}$ and $^{76}\text{Se}$, compared with the experimental occupancies of valence orbits [6, 7].

GCM Spectrum in 2 Shells

ββ Matrix Elements in 1 and 2 Shells

$^{76}\text{Ge}$ $^{76}\text{Se}$

Excitation energy (MeV)

0 1 2 3 4
Exp. GCM

$0^+_1$ $0^+_2$ $0^+_3$ $0^+_3$

ββ Matrix Elements in 1 and 2 Shells

$M_{GT}$

$^{48}\text{Ca}$ $^{48}\text{Ca}$ $^{76}\text{Ge}$ $^{76}\text{Ge}$ $^{82}\text{Se}$ $^{76}\text{Ge}$ $^{76}\text{Ge}$

SM GCM

JUN45 GCN2850 GCN2850 pfsdg-1 pfsdg-2

KB3G SDPFMU-DB
Combining GCM and Ab Initio Methods

GCM incorporates some correlations that are hard to capture automatically (e.g. shape coexistence). So use it to construct initial “reference” state, let IMSRG, do the rest.

Energy evolution in single shell

Ab initio $0\nu\beta\beta$ matrix element in mirror pair

From J. Yao

In progress: Matrix elements for detector nuclei.
Could be considerably harder.
Improving RPA/QRPA

Upcoming talks by J. Terasaki, C. Robin

RPA produces states in intermediate nucleus, but form is restricted to 1p-1h excitations of ground state. Second RPA adds 2p-2h states.
Issue Facing All Models: “$g_A$”

40-Year-Old Problem: Effective $g_A$ needed for single-beta and two-neutrino double-beta decay in shell model and QRPA.

If $0\nu$ matrix elements quenched by same amount as $2\nu$ matrix elements, experiments will be much less sensitive; rates go like fourth power of $g_A$.

Upcoming talks by S. Pastore, L. Coraggio?
Arguments Suggesting Strong Quenching of $0^\nu$

- Both $\beta$ and $2\nu\beta\beta$ rates are strongly quenched, by consistent factors.
- Forbidden ($2^-$) decay among low-lying states appears to exhibit similar quenching.
- Quenching due to correlations shows weak momentum dependence in low-order perturbation theory.
Arguments Suggesting Weak Quenching of 0\(\nu\)

- Many-body currents seem to suppress 2\(\nu\) more than 0\(\nu\).
- Enlarging shell model space to include some effects of high-\(j\) spin-orbit partners reduces 2\(\nu\) more than 0\(\nu\).
- Neutron-proton pairing, related to spin-orbit partners and investigated pretty carefully, suppresses 2\(\nu\) more than 0\(\nu\).

![Graph showing Ca → Ti transitions for 0\(\nu\) and 2\(\nu\) with various values of \(g_{pp}\).]

Large \(r\) contributes more to 2\(\nu\).
Problem must be due to some combination of:

1. **Truncation of model space.**
   Should be fixable in ab-initio shell model, which compensates effects of truncation via effective operators.

2. **Many-body weak currents.**
   Size still not clear, particularly for $0\nu\beta\beta$ decay, where current is needed at finite momentum transfer $q$.
   
   Leading terms in chiral EFT for finite $q$ only recently worked out. Careful fits and use in decay computations will happen in next year or two.
Finally...

- Also coming…Talks on neutrino oscillations by S. Bilenky and double charge exchange by N. Auerbach.

- Topical collaboration will speed progress in next few years. Or else …I don’t want to think about it.

That’s all.
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