Description of few-nucleon collisions
by FY eq’s approach

R. Lazauskas
• Solution of the 3-5 body Faddeev-Yakubovsky equations for nuclear systems

• Application of the complex-scaling method to solve complicated scattering problems using trivial boundary conditions
Collisions

• In configuration space wave functions extend to infinity!

• Increasingly complex asymptotic behaviour for $A>2$ systems!!

How to take care of the boundary condition?

✓ Conceptual difficulties to uncouple different particle channel, to constrain asymptotes of the solutions in all directions and thus get unique (physical) solution to the Schrodinger eq.

• It is ok, as long as there is single particle channel (elastic plus target excitations)

• Mathematically ill-conditioned problem when several particle channels are open

✓ Faddeev-Yakubovsky equations efficiently separates asymptotes of the binary channels

Equations for short-ranged pairwise interactions

3-body
(Faddeev eq.)

\[ \phi_{12} = G_0 V_{12} \Psi \]

\[ \phi_{23} = G_0 V_{23} \Psi \]

\[ \phi_{31} = G_0 V_{31} \Psi \]

\[ \Psi = \phi_{12} + \phi_{23} + \phi_{31} \]

4-body
(Faddeev-Yakubovskiy eq.)

\[ \phi_{jk} = G_0 V_{jk} \Psi \]

\[ K_{ij,k} = G_{ij} V_{ij} (\phi_{jk} + \phi_{ik}); \quad H_{ij}^{kl} = G_{ij} V_{ij} \phi_{kl} \]

\[ \Psi = \sum_{i<j} \phi_{ij} = \sum_{i<j} (K_{ij,k}^l + K_{ij,l}^k + H_{ij}^{kl}) \]
\[(E - H_0 - V_{12}) K_{12,3}^{4} = V_{12} (K_{13,2}^{4} + K_{23,1}^{4} + K_{13,4}^{5} + K_{23,4}^{5} + K_{13,4}^{2} + K_{23,4}^{2} + T_{13,4} + T_{23,4} + H_{13}^{24} + H_{23}^{14} + S_{13}^{24} + S_{23}^{14} + F_{13}^{24} + F_{23}^{14}) \]

\[(E - H_0 - V_{12}) H_{12}^{34} = V_{12} (H_{34}^{12} + H_{34,1}^{2} + H_{34,2}^{1} + H_{34,1}^{5} + H_{34,2}^{5} + T_{34,1} + T_{34,2}) \]

\[(E - H_0 - V_{12}) T_{12,3} = V_{12} (T_{13,2} + T_{23,1} + H_{13}^{45} + H_{23}^{45} + S_{13}^{45} + S_{23}^{45} + F_{13}^{45} + F_{23}^{45}) \]

\[(E - H_0 - V_{12}) S_{12}^{34} = V_{12} (F_{34}^{12} + S_{34}^{15} + S_{34}^{25} + F_{34}^{15} + F_{34}^{25} + H_{34}^{15} + H_{34}^{25}) \]

\[(E - H_0 - V_{12}) F_{12}^{34} = V_{12} (S_{34}^{12} + K_{34,5}^{1} + K_{34,5}^{2} + T_{34,5}) \]
Merits:

✓ Handling of symmetries
✓ Boundary conditions for binary channels
✓ Easy reduction to subsystems

Price

✓ Overcomplexity

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<tr>
<th>Problem</th>
<th>Number eq. (identical particles)</th>
<th>Number eq. (different particles)</th>
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<tr>
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<td>18</td>
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<tr>
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<td>A=6</td>
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<tr>
<td>A=N</td>
<td></td>
<td>(\frac{N! (N-1)!}{2^{N-1}})</td>
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R. Lazauskas
5-body Faddeev-Yakubovski eq

\[ K_{12,3}^4 \left( \vec{x}, \vec{y}, \vec{z}, \vec{w}, S, L, T \right) = \sum_{\alpha_K = (l, s, t, \ldots)} f_{\alpha_K}(x, y, z, w) \frac{1}{xyzw} \left\{ \left( l_x l_y \right)_L \left( l_z l_w \right)_T \right\}_L \{ \ldots \}_S \{ \ldots \}_T \]

**NUMERICAL SOLUTION**


- PW decomposition of the components K,H,T,S,F
- Radial parts expanded using Lagrange-mesh method
- Resulting linear algebra problem solved using iterative methods
- Observables extracted using integral relations

D. Baye, Physics Reports 565 (2015) 1

R. Lazauskas
**Problem Number**

<table>
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<th>Number eq. (diff. particles)</th>
<th>PW basis.</th>
<th>Radial disc.</th>
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<td>2</td>
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<td>~10^6</td>
<td>~N^4</td>
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**NUMERICAL SOLUTION**


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Short overview of nuclear problems by FY eq’s

3N-problem (Faddeev)

1st solution: A. Laverne and C. Gignoux


4N-problem (Faddeev)

1st solution:

(1984) 125

Benchmarks:


M. Viviani et al., Phys. Rev. C 84 (2011) 054010 (p3He scattering)

M. Viviani et al., arXiv:1610.09140 (p3H,n3He scattering)

\[ p + ^3 H \leftrightarrow n + ^3 He \]

FIG. 5: (Color online) Differential cross sections (upper panels) and proton analyzing powers \( A_{y0} \) (lower panels) for \( p ^3 H \) elastic scattering at \( E_p = 2.5, 3.5, \) and 4.15 MeV proton energies obtained using the N3LO500 potential. The lines show the results obtained using the AGS (blue solid lines), FY (red dot-dash lines), and the HII (green dashed lines) methods. In many cases, the curves overlap and cannot be distinguished. The experimental data in panel (a) are from Refs. [36] (circles) and [35] (squares), in panel (b) from Refs. [36] (circles), [39] (squares), and [40] (triangles), in panel (c) from Refs. [41] (circles) and [40] (squares), and finally in panel (f) from Ref. [41] (circles).
5N problem: $n^{-4}\text{He}$ scattering

MT I-III S-wave potential

\[ \delta (\circ) \]
\[ E_{\text{cm}}^4 \text{ (MeV)} \]

- $^2\text{P}_{3/2}$
- $^2\text{P}_{1/2}$
- $^2\text{S}_{1/2}$

- No Coulomb
- + Coulomb
n-$^4$He scattering


R. Lazauskas
n-$^3$H total cross section

\[ \sigma(E) \]

\( E \text{ (MeV)} \)

\( \sigma \text{ (b)} \)

- **MT I-III**
- **Av.18**
- **Av.18+UIX**
- **INOY**

\( E \text{ (MeV)} \)

\( \sigma \text{ (b)} \)

- **I-N3LO+3BF(N2LO)**
- **I-N3LO**
- **NLO**
- **Av.18+UIX**
- **INOY**
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What to do?

Take care of the boundary condition

- Faddeev-Yakubovsky equations efficiently separates asymptotes of the binary channels

BUT

- Number of the reaction channels increases rapidly with $A$
- Even for 3-body systems there may exist infinite number of binary channels $e + H \rightarrow e + H (n = 1, 2, ..., \infty)$
- Complex behavior of the breakup asymptotes

FIND SOME TRICKS TO AVOID PROBLEMS AT BOUNDARY!
A dream: to solve scattering problems with a similar ease as bound state one (avoiding complex singularities or boundary conditions)

A. Deltuva, R.L. et al., PPNP 74 (2014)


In Quantum-Mechanics problem of $N$-particle dynamics may be often formulated in time-independent formalism and for the wave functions whose asymptotes contain only nontrivial outgoing waves

$$\left( E_{sc} - H_0 - \sum_{i>j} V_{ij}(r_{ij}) \right) \psi(r_i) = S(r_i)$$

- **Bound state problem:** $S(r_i) \equiv 0$
- **Resonances:** $S(r_i) \equiv 0$
- **Reactions due to external probe:** $S(\rho \to \infty) = 0$
- **Collisions:** $S(r_i) \equiv 0$

$$\psi(\rho \to \infty) \propto e^{ik_{x}r_{x}}$$

$$\psi_{sc}(\rho \to \infty) \propto \psi_{in}(r_i) + \sum_{c} A_{c}(\hat{k}_{c}) e^{i|k_{c}|r_{c}}$$

R. Lazauskas
Complex scaling: resonances, reactions & bound states in an unified formalism

R. Hartree, J.G. L. Michel, P. Nicolson (1946)

Complex scaling kills outgoing waves in the asymptote

\[ r \rightarrow r e^{i\theta} \]

\[ \tilde{\psi}(\rho \rightarrow \infty) \propto e^{i k x r e^{i\theta}} = e^{i k x r \cos \theta} e^{-k x r \sin \theta} \]
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\]

\[
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\]

Diverges

R. Lazauskas
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exp. bound if \( 0 < \theta < \pi \)

- Schrödinger equation in a driven form:

\[ r \Psi(r) = F_l^{\text{in}}(r) + F_l^{\text{sc}}(r) \]

\[ \frac{\hbar^2}{2\mu} \left( -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V_l(r) - k^2 \right) F_l^{\text{sc}}(r) = -V_l(r) F_l^{\text{in}}(r) = 0 \]

Complex scaling \( r \rightarrow r e^{i\theta} \)

\[ \frac{\hbar^2}{2\mu} \left( -\frac{d^2}{e^{2i\theta} dr^2} + \frac{l(l+1)}{e^{2i\theta} r^2} + V_l(re^{i\theta}) - k^2 \right) \tilde{F}_l^{\text{sc}}(r) = -V_l(re^{i\theta}) F_l^{\text{in}}(re^{i\theta}) \]

exp. bound for the short range pot. term \( V_s \)
Complex scaling: guide for pedestrians

Solution (using standard bound state techniques)

1. Expand c.s. outgoing wave in your favorite basis
   \[ \tilde{\Psi}^{sc}(r) \approx \sum_{i=1}^{N} c_i \psi_i(r) \]

2. Convert c.s. Schrödinger equation into linear algebra problem

3. \[
   \int \psi_j(r) dr : \quad \frac{n^2}{2\mu} \left( -\frac{d^2}{e^{2i\theta} dr^2} + \frac{l(l+1)}{e^{2i\theta} r^2} + V_l(re^{i\theta}) - k^2 \right) \tilde{\Psi}_i^{sc}(r) = -V^*_l(re^{i\theta})\tilde{\Psi}_i^{so}(re^{i\theta})
   \]

4. Solve linear algebra problem to determine coefficients \( c_i \)
      limit \( 10^5 \) basis st.
   ii. (Iterative) linear algebra methods
      upto \( 10^{10} \) basis st.

5. Extract scattering observables from obtained solution(s): multiple choice, most efficient way via integral expressions
Review of the method*

- Complex scaling method is very functional, adaptable to almost any b.s. technique. Already successfully applied using:
  - Spline basis
  - Laguerre, HO, Gaussian, correlated Gaussian, Sturmian basis functions
  - Lagrange mesh method


- External probes
  - 2-body, 3-body, 4-body systems, including repulsive Coulomb


- Collisions, demonstrated to work for:
  - 2-body collisions including Coulomb interaction, Optical potentials, $\frac{1}{r^n}$ potentials with $n \geq 4$
  - 3-body scattering including the break-up
  - 3-body scattering including non-local, Optical potential, attractive/repulsive Coulomb interactions
  - 3-body break-up amplitude for n-d & p-d scattering
  - 4-body scattering in 4N systems, including Coulomb


R. Lazauskas
n-d scattering for MT I-III potential

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<tr>
<th></th>
<th>0°</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
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<tr>
<td>Ref. [23] Im($^3S_1$)</td>
<td>1.69[0]</td>
<td>1.74[0]</td>
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<td>1.95[0]</td>
<td>2.52[0]</td>
<td>3.06[0]</td>
</tr>
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$^2\text{H} + ^{12}\text{C} \leftrightarrow p + ^{13}\text{C}$ scattering within 3-body model

Optical CH89 pot.

$^2\text{H}$ $^{12}\text{C}$ $p$ $n$

$^{12}\text{C} + d \leftrightarrow ^{13}\text{C} + p$

$^{12}\text{C} + n + p$

$E_{cm}$ (MeV) 28.4 30.6

$E_p$ (MeV) 0 2.7 4.9

p-^3^He scattering/realistic NN interactions

\[ d\sigma/d\Omega \text{ (mb/sr)} \]

\[ \theta_{\text{c.m.}} \]

\[ E_p = 30 \text{ MeV} \]

\[ 0 \text{ } 30 \text{ } 60 \text{ } 90 \text{ } 120 \text{ } 150 \text{ } 180 \]

[Graph showing the angular distribution of differential cross section with theta angles from 0 to 180 degrees.


\[ p+^3^He \]

\[ p+p+^2^H \]

\[ p+p+n+p \]

\[ 0 \] 5.5 7.7

\[ E_{\text{cm}} \text{ (MeV)} \]

\[ 22.5 \]

\[ 30.0 \]

\[ E_p \text{ (MeV)} \]
• C.S. outgoing waves asymptotically converges as $e^{-|k_{cp}|rsin(\theta)}$
  ♦ Slow convergence for small $k_{cp}$, difficulties in close-threshold region
  😊 Convergence is faster for large $\theta$
• One should work with c.s. potential or c.s. basis functions
  ♦ Imposes upper limit for the $q$ to be used, since

$$V(r) \sim \exp(-\mu r^n) \rightarrow \exp(-\mu r^n e^{in\theta})$$

→ Starts to diverge for $\theta > \pi/(2n)$

![Graphs showing potential functions for different angles](image)
• FY eq. formalism remains reference in few-body scattering calculations. Four-nucleon scattering problem approaches fixed status. The first solution of 5-body FY equations is presented.

• At the same time several promising methods emerge to solve scattering problems based on trivial boundary conditions.

• In particular, complex-scaling method has been revived in nuclear physics. This method enables to solve few-body scattering problem employing standard bound state techniques (feasible by almost any config. space bound state technique and requires very limited effort to be implemented).

• Simple extension of the formalism to many-body scattering case.

• Very accurate results are already obtained for 3-body and 4-body elastic and breakup scattering.

Acknowledgements: The numerical calculations have been performed at IDRIS (CNRS, France). We thank the staff members of the IDRIS computer center for their constant help.