Recent Developments in Numerical Relativity

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What is numerical relativity

**Numerical Relativity:**

solving numerically the full GR equations, typically for dynamical spacetimes in the strong field regime, where no approximations hold.

**Goals:**

understanding gravity in its full non-linear glory.

**Challenges:**

very difficult problem...
Why numerical relativity

Study of systems with strong and dynamical gravitational fields

- Gravitational radiation
  - Astrophysics, gravitational wave astronomy
- Mathematical and theoretical Physics
  - Cosmic censorship,
  - Instabilities (Black hole interior, Myers-Perry)
- High-energy particle systems
  - AdS/CFT correspondence;
  - Black hole production at the LHC;
Gravitational waves

- Accelerated bodies emit gravitational radiation
- Detected indirectly by measurements of the Hulse-Taylor binary system (1993 Nobel Prize)
- Interact weakly with matter ⇒ carry unique information about astronomical phenomena
  - ⇒ New window to the universe
Gravitational waves

- Difficult to detect
- ⇒ Need theoretical models for the structure of the waveform
Mathematical and theoretical Physics

- Cosmic censorship hypothesis:
  - does it hold under extreme conditions?
    Sperhake, Cardoso, Pretorius, Berti, Gonzales, 2008

- No no-hair theorem for $D > 4 \Rightarrow$ black hole solutions with non-spherical topology.
  - (Non-)Linear stability of higher-dimensional black objects:
    - Black string
      Choptuik, Lehner, Olabarrieta, Petryk, Pretorius, Villegas, 2003
      Lehner, Pretorius 2010
    - Myers-Perry black hole
      Shibata & Yoshino, 2010
    - Black ring
    - ...

Emparan & Reall, 2008
**AdS/CFT duality**

- Properties of strongly coupled thermal gauge theories are related to the physics of higher-dimensional black holes.
- Formation of quark-gluon plasma at the RHIC $\Leftrightarrow$ black hole collisions in AdS$_5$.

**Issues with numerical simulations in AdS:**
- AdS is not globally hyperbolic.
- The boundary plays an active role.
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# History and milestones

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<th>Event</th>
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<tr>
<td>1915</td>
<td>Einstein’s equations are published</td>
<td>Einstein</td>
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<tr>
<td>1964</td>
<td>First documented attempts at numerical simulations: evolving two wormholes</td>
<td>Hahn &amp; Lindquist</td>
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<td>1976</td>
<td>Head-on collision of two black holes (in axisymmetry)</td>
<td>Smarr &amp; Eppley</td>
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<td>1990's</td>
<td>&quot;Binary Black Hole Grand Challenge Project&quot;</td>
<td>Matzner et al</td>
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<td>1993</td>
<td>Critical phenomena in gravitational collapse</td>
<td>Choptuik</td>
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<td>1997</td>
<td>Release of Cactus 1.0</td>
<td>Seidel et al</td>
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<td>1998</td>
<td>Generic (3D) single BH simulation (using a characteristic approach)</td>
<td>Gomez et al</td>
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<td>1999</td>
<td>BSSN evolution system</td>
<td>Baumgarte &amp; Shapiro; Shibata &amp; Nakamura</td>
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<td>2005</td>
<td>First simulations of BH binaries through inspiral, merger and ringdown (Two-body problem in GR) (GHG code)</td>
<td>Pretorius</td>
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<td>2006</td>
<td>&quot;Moving puncture&quot; simulations (BSSN code)</td>
<td>UTB/RIT; NASA Goddard</td>
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<td>2008</td>
<td>High-energy collision of two BHs</td>
<td>Berti, Cardoso, Gonzalez, Sperhake, Pretorius</td>
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<td>2010</td>
<td>Collision of gravitational shock waves in AAdS5 spacetimes (2+1 code)</td>
<td>Chesler &amp; Yaffe</td>
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<td>2010</td>
<td>Black hole collisions in higher dimensions</td>
<td>Witek, M.Z. et al; Yoshino &amp; Shibata</td>
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<td>2012</td>
<td>Simulations of AAdS5 spacetimes (GHG code)</td>
<td>Bantilan, Pretorius, Gubser</td>
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<td>2015</td>
<td>Off-center collisions of shock waves in AAdS5 spacetimes (4+1 code)</td>
<td>Chesler &amp;Yaffe</td>
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Einstein’s equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi \, T_{\mu\nu} \]
Einstein’s equations

\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \sum_{\delta=t,x^1,\ldots,x^{D-1}} g^{\alpha\delta} \left( \partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma} \right)
\]

\[
8\pi T_{\alpha\beta} = \sum_{\delta} \left[ \partial_\delta \Gamma^\delta_{\alpha\beta} - \partial_\alpha \Gamma^\delta_{\delta\beta} + \sum_\gamma (\Gamma^\delta_{\alpha\beta} \Gamma^\gamma_{\delta\gamma} - \Gamma^\delta_{\gamma\beta} \Gamma^\gamma_{\delta\alpha}) \right]
\]

\[
-\frac{1}{2} g_{\alpha\beta} \sum_{\delta,\gamma} \left\{ g^{\delta\gamma} \sum_\mu \left[ \partial_\mu \Gamma^\mu_{\delta\gamma} - \partial_\delta \Gamma^\mu_{\mu\delta} + \sum_\nu (\Gamma^\mu_{\delta\gamma} \Gamma^\nu_{\mu\nu} - \Gamma^\mu_{\nu\gamma} \Gamma^\nu_{\mu\delta}) \right] \right\}
\]
Before numerical evolution...

- Write Einstein’s equations as a well-posed Initial Boundary Value Problem (IBVP):
  - solution’s behaviour depends continuously with the initial data;
  - numerically suitable gauge conditions.
- Discretize resulting PDEs
- Specify constraint preserving, and physically correct, boundary conditions
- Find a way to deal with singularities
During numerical evolution...

- Compute constraint-satisfying initial data representing snapshot of physical system
- Implement mesh refinement, or similar, to efficiently handle different length scales (and parallelize resulting algorithms)
- Extract physical results in gauge-invariant fashion from numerical data
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3+1 decomposition

We write the metric as

\[ ds^2 = -\alpha^2 dt^2 + \gamma_{ij} \left( dx^i + \beta^i dt \right) \left( dx^j + \beta^j dt \right), \]

- \( \gamma_{ij} \) is the metric on surfaces of \( t = \text{const} \)
- \( K_{ij} \) is the extrinsic curvature
ADM-York evolution equations

Evolution equations

\[(\partial_t - L_\beta) \gamma_{ij} = -2\alpha K_{ij},\]
\[(\partial_t - L_\beta) K_{ij} = -\nabla_i \nabla_j \alpha + \alpha \left[ R_{ij} + KK_{ij} - 2K_{ik}K^k_j \right.\]
\[\left. + \frac{8\pi}{D-2} \left( (S - E)\gamma_{ij} - 2S_{ij} \right) \right],\]

Constraints

\[R + K^2 - K_{ij}K^{ij} = 16\pi E,\]
\[\nabla_j \left( K^{ij} - \gamma^{ij} K \right) = 8\pi p^i.\]
Electromagnetic analogy

Evolution equations

\[-\partial_t \vec{E} + \nabla \times \vec{H} = 4\pi \vec{j}\]
\[-\partial_t \vec{H} + \nabla \times \vec{E} = 0\]

Constraints

\[\nabla \cdot \vec{E} = 4\pi \rho\]
\[\nabla \cdot \vec{H} = 0\]
Well-posedness

Consider evolution equations of the form

$$\partial_t \phi + M^i \partial_i \phi = S(\phi)$$

The system is well-posed iff

$$\| \phi(t, x^i) \| \leq ke^{\alpha t} \| \phi(0, x^i) \|$$

ie, the norm of the solution can be bounded by the same exponential for all initial data.
Construct matrix $P(n_i) = n_i M^i$, for arbitrary unit vector $n^i$ (principal symbol).

Strongly hyperbolic system:
if $P$ has real eigenvalues and complete set of eigenvectors for all $n^i$.

Weakly hyperbolic system:
if $P$ has real eigenvalues but does not have a complete set of eigenvectors.

If a system is strongly hyperbolic, it is well-posed.
BSSN equations

... the ADM equations are only weakly hyperbolic...

- Alternative formulations to ADM system started being explored in the late 1980s, before full impact of the hyperbolicity properties of the different formulations had been realized.

- Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation is derived from the ADM equations but works with conformally rescaled variables, a trace split of the extrinsic curvature and promotes the contracted Christoffel symbols to the status of independent variables.
BSSN equations

\[\partial_t \tilde{\gamma}_{ij} = \beta^k \partial_k \tilde{\gamma}_{ij} + 2\tilde{\gamma}_k (i \partial_j) \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k - 2\alpha \tilde{A}_{ij},\]

\[\partial_t \chi = \beta^k \partial_k \chi + \frac{2}{3} \chi (\alpha K - \partial_k \beta^k),\]

\[\partial_t \tilde{A}_{ij} = \beta^k \partial_k \tilde{A}_{ij} + 2\tilde{A}_k (i \partial_j) \beta^k - \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k + \chi (\alpha R_{ij} - \nabla_i \partial_j \alpha)^TF\]

\[+ \alpha \left(K \, \tilde{A}_{ij} - 2\tilde{A}_i^k \tilde{A}_{kj}\right) - 8\pi \alpha \left(\chi S_{ij} - \frac{S}{3} \tilde{\gamma}_{ij}\right),\]

\[\partial_t K = \beta^k \partial_k K - \nabla^k \partial_k \alpha + \alpha \left(\tilde{A}^i_j \tilde{A}_{ij} + \frac{1}{3} K^2\right) + 4\pi \alpha (E + S),\]

\[\partial_t \tilde{\Gamma}^i = \beta^k \partial_k \tilde{\Gamma}^i - \tilde{\Gamma}^k \partial_k \beta^i + \frac{2}{3} \tilde{\Gamma}^i \partial_k \beta^k + 2\alpha \tilde{\Gamma}^i_j k \tilde{A}^j k + \frac{1}{3} \tilde{\gamma}^{ij} \partial_j \partial_k \beta^k\]

\[+ \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i - \frac{4}{3} \alpha \tilde{\gamma}^{ij} \partial_j K - \tilde{\gamma}^{ij} \left(3\alpha \chi^{-1} \partial_j \chi + 2\partial_j \alpha\right) - 16\pi \alpha \chi^{-1} j^i\]
Moving Punctures

This system, together with the moving puncture gauge conditions, allowed for the 2005-06 breakthrough simulations of the Brownsville/RIT and NASA-Goddard groups.
Kick configuration

Simulation:
  Manuela Campanlli
  Carlos Lousto
  Yosef Zlochower

Visualization:
  Hans-Peter Bischof

CCRG
RIT

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Generalized Harmonic Gauge (GHG)

Imposing coordinates satisfying condition (harmonic coordinates)

\[ H^\alpha \equiv \Box x^\alpha = -g^{\mu\nu} \Gamma_{\mu\nu}^\alpha = 0 \]

and promoting these to independently evolved variables, the (generalized) Einstein equations take the form:

\[
g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} = -2 \partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - 2 \partial_{(\alpha} H_{\beta)} + 2 H_\mu \Gamma_{\alpha\beta}^\mu - 2 \Gamma_{\nu\alpha}^\mu \Gamma_{\mu\beta}^\nu - 8\pi T_{\alpha\beta} + 4\pi T g_{\alpha\beta} - 2\kappa \left[ 2n_{(\alpha} C_{\beta)} - \lambda g_{\alpha\beta} n^\mu C_\mu \right]
\]
Generalized Harmonic Gauge (GHG)

- Set of 2nd order wave equations for each component of spacetime metric
- Equations are symmetric hyperbolic
- Used in Pretorius’ 2005 breakthrough simulations
Black Hole Inspiral

F. Pretorius
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Characteristic Initial Value Problem

\[ ds^2 = - \left( e^{2\beta} \frac{V}{r} - r^2 h_{AB} U^A U^B \right) du^2 \]
\[ - e^{2\beta} dudr - 2r^2 h_{AB} U^B dudx^A + r^2 h_{AB} dx^A dx^B \]

Schematic evolution equations:

\[ \partial_r F = H_F(F, G) \]
\[ \partial_u \partial_r G = H_G(F, G, \partial_u G) \]
Advantages of characteristic evolution

- Initial data is free (no elliptic constraints on the data);
- No second time derivatives (therefore smaller number of basic variables);
- Rigorous description of gravitational waves on null hypersurfaces (ideal for wave-extraction);
- Equations have convenient hierarchical structure in which variables are integrated in turn in terms of characteristic data from prior members of the hierarchy.

First stable 3D evolutions of black holes (including moving and rotating configurations) were achieved with characteristic methods in 1997, which proved remarkably stable.
Formalism

Characteristic-based approach

Toy problem (wave equation)

Cauchy:

\[ \partial_{tt}^2 \phi - \partial_{xx}^2 \phi = 0 \]

Characteristic \((u = t - x, x = r)\):

\[ 2\partial_u \partial_r \phi - \partial_{rr}^2 \phi = 0 \]

- solution: \( \phi(u, r) = F(u) + G(u + 2r) \).
  - \( F \) – outgoing wave; \( G \) – ingoing wave
- Solved provided \( \phi(u = u_0, r) \) and boundary data at \( r = 0 \) are given.
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Why Cactus/ET

Typical problem in Numerical Relativity...
- mesh refinement
- efficiently parallelize
- large input/output
- somewhat complex tools for analysis

Typical workflow:
1. Compute initial data
2. Evolve equations
3. Analysis
What is Cactus

Cactus:

- general framework for the development of portable, modular applications
- programs are split into independent components (thorns)
- thorns are developed independently and should be interchangeable with others with same functionality
- thorns don’t directly interact with each other
- Cactus framework (flesh) provides the “glue”
- supports C, C++, Fortran
Obtaining ET

Welcome

The Einstein Toolkit Consortium is developing and supporting open software for relativistic astrophysics. Our aim is to provide the core computational tools that can enable new science, broaden our community, facilitate interdisciplinary research and take advantage of emerging petascale computers and advanced cyberinfrastructure.

Please read our pages about the Einstein Toolkit, its governance, and how to get started with the toolkit for more information.

Download

November 2012: We are pleased to announce the sixth release (code name "Ørsted") of the Einstein Toolkit, an open, community developed software infrastructure for relativistic astrophysics.
Obtaining ET

**DOWNLOAD**

The Einstein Toolkit is hosted on many different machines around the world. We provide a script called `GetComponents` to simplify downloading the toolkit. This page just describes how to download the toolkit - you may also be interested in the [Tutorial for New Users](#) which leads you through these steps and more on the Queen Bee supercomputer.

Enter the directory on your machine in which you would like to download the ET (for example, your home directory), and type the commands listed below. This will create a directory called Cactus in which the components of the Einstein Toolkit are downloaded.

**Current release: Ørsted (released on November 8th, 2012)**

This is the recommended version of the toolkit for most users. See the [release notes](#) for more information.

```
  curl -O https://raw.githubusercontent.com/gridaphobe/CRL/master/GetComponents
  chmod a+x GetComponents
```
ET contents: arrangements

Provided arrangements (≈ collection of thorns):
- Several Cactus thorns (I/O, Method of Lines, . . .)
- Carpet (Adaptive Mesh Refinement driver)
- EinsteinBase
- EinsteinInitial
- EinsteinEvolve
- EinsteinAnalysis
- McLachlan (BSSN implementation)
- . . .
Cactus arrangements

Main core Cactus arrangements:

**CactusBase**
Infrastructure thorns for boundary conditions, coordinates, IO, symmetries and time

**CactusNumerical**
Numerical infrastructure thorns: time integration, dissipation, symmetry boundary conditions, spherical surfaces, local interpolation, Method of Lines (MoL), . . .

**CactusUtils**
Utility thorns: formaline, nan-checking, termination triggering and timer reports

**ExternalLibraries**
Provides external libraries: Lapack, GSL, HDF5, FFTW, Lorene (for initial data) and others
Carpet

Berger-Oliger Adaptive Mesh Refinement (AMR) driver:
- Splits grid functions and arrays among the MPI processes
- Setups mesh refinement grid hierarchy
- Communicates ghost cell information between MPI processes
- Communicates between refinement levels by prolongation and restriction
- Modifies grid hierarchy (regridding) when requested
- Performs parallel IO
ET Arrangements

**EinsteinBase**
- defines basic spacetime variables

**EinsteinInitial**
- computing initial data

**EinsteinEvolve**
- evolves variables in time (typically with Method Of Lines)

**EinsteinAnalysis**
- diagnostic tools
Demo
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Miguel Zilhão (UB)
Recent Developments in Numerical Relativity
INT-15-2c, 17 Aug 45 / 57
Recent developments in Numerical Relativity

Higher-dimensional BH collisions

Axial symmetry $SO(D - 2)$ and $SO(D - 3)$

- Highly symmetric systems;
- Can be reduced to effective $3 + 1$ systems;
  ⇒ We can use existing numerical codes (with adaptations);
**D = 5 head-on collision from rest**  
*Witek, MZ et al 2010*

characteristic ringdown frequency:

\[
 r_S \omega = 0.955 \pm 0.005 - i(0.255 \pm 0.005) \\
(r_S \omega = 0.9477 - i0.2561, \text{ e.g., Berti et al. 2009})
\]
Black hole collisions in $D = 4$

- two black holes
  - Total rest mass: $M_0 = M_A + M_B$
  - Initial position: $\pm x_0$
  - Linear momentum: $\mp P[\cos \alpha, \sin \alpha, 0]$
- Impact parameter: $b \equiv \frac{L}{P}$
Recent developments

**Head-on collisions:** \( b = 0, \quad \vec{S} = 0 \)

- **Total radiated energy:** \( 14 \pm 3 \% \) for \( v \rightarrow 1 \) *Sperhake et al. 2008*

![Graph showing the total radiated energy as a function of \( \beta \).]

- **Agreement with approximative methods** *Berti et al. 2010*
- **Recently revisited with improved initial data by RIT group** *Ruchlin et al 2015*
$D = 4$ Boosted collisions  
Sperhake et al 2008

U. Sperhake
Grazing collisions

- Two distinct end-states to BH scattering problem: one BH or two BHs
- Near the critical impact parameter:
  - sensitivity to initial conditions
  - enhanced gravitational wave emission (even in scattering cases)
Whirl, merger

U. Sperhake
Black String

Class of higher-dimensional black hole solutions

\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2 + dw^2 \]

- Shown to be unstable to long-wavelength perturbations
  - Gregory & Laflamme 1993

- A lot of debate about the end-state
Recent developments

- BH Collisions in AAdS spacetimes  
  Bantilan & Romatschke 2015

- Off-center collisions of shock waves in AAdS5 spacetimes  
  Chesler & Yaffe

- GRChombo: NR with Adaptive Mesh Refinement  
  Clough, Figueras et al 2015

- Accretion disks around binary BHs of unequal mass: General relativistic MHD simulations  
  Gold et al 2014

- …
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Final Remarks

- Numerical modelling of gravitational systems is as old as the advent of computing itself.
- 2005 breakthroughs in NR marked a phase transition in the field.
- This allowed for the discovery of unexpected results (superkicks of thousands of km/s, zoom-whirl behaviour, etc.)
- Motivation for long-term NR efforts came originally mostly from the modeling of gravitation wave sources.
- Nowadays, NR is finding applications to other fields, such as high-energy physics, higher-dimensional gravity, and AdS/CFT.
- What surprises will next 10 years of NR reveal?