Nonlocal Wave Turbulence in QCD

(In preparation)

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Wave Turbulence (1)

- Out-of-equilibrium statistics of random non-linear waves

- Similarity with fluid turbulence: inviscid transport of conserved quantities from large to small scales through the so-called transparency window or inertial range

- Examples:
  - Atmospheric Rossby waves
  - Water surface gravity and capillary waves
  - Waves in plasmas
  - Nonlinear Schrödinger equation (NL Optics, BEC)
Wave Turbulence ( II )

- Waves are excited by external processes. Driven turbulence: Open system with source and sink → away from thermodynamical equilibrium

- Steady states characterized by constant fluxes P and Q rather than temperature and thermodynamical potentials

- Kolmogorov-Obukhov (KO41) theory relies on Locality of interactions: Only eddies (waves) with comparable sizes (wavelengths) interact. Steady state power spectra in momentum space depend on the fluxes and not on the pumping and dissipation scales

- Weak (Wave) Turbulence Theory: Kinetic description in the limit of weak nonlinearities (No theory for strong turbulence)

V. E. Zakharov, V. S. L’vov, G. Falkovich (Springer-Verlag, 1992)
Turbulence in early stages of Heavy Ion Collisions

- Far-from equilibrium evolution of anisotropic particle distribution in momentum space (Initial conditions in HIC)

- Chromo-Weibel Instabilities: Anisotropy (hard modes) induces exponentially growing soft modes - transverse magnetic and electric fields which help restoring isotropy - (early stage: as in abelian plasmas) turning to a linear growth due to nonlinear interactions inherent to non-abelian plasmas

E. S. Weibel (1959)
S. Mrowczynski (1993)
D. Bödeker, K. Rummukainen (2005)
Turbulence in early stages of Heavy Ion Collisions

- Hard-loop simulations (large scale separation between hard modes and soft excitations): Nonlinear interactions develop a **turbulent cascade** in the UV with exponent \( \nu = 2 \)

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**P.Arnold, G. D. Moore (2005)**

Weak Turbulence in Kinetic Theory

- Can one understand this power spectrum $k^{-2}$ from first principles?

- **Note:** from A. H. Mueller, A. I. Shoshi, S. M. H. Wong (2006):

  \[
  \text{Turbulence in QCD is nonlocal } \Rightarrow \quad n(k) \sim k^{-1}
  \]

- **Caveats (in this work):**
  - Homogeneous and isotropic system of gluons
  - Energy injection with constant rate $P$ at $k_f \gg m$ : Dispersion relation $\omega(k) \equiv |k|
  - Weak nonlinearities in the classical limit (high occupancy):
    \[
    g^2 \ll 1 \quad \text{and} \quad 1 \ll n(k) \ll \frac{1}{g^2}
    \]
Elastic 2 to 2 process (4-waves interactions)

- Elastic gluon-gluon scattering

\[
\frac{\partial}{\partial t} n_k = \frac{1}{2} \int_{k_1, k_2, k_3} \frac{1}{2\omega(k)} |M_{12 \rightarrow 3k}|^2 \delta(\sum_i k_i) \delta(\sum_i \omega_i) F[n]
\]

\[
F[n] \equiv [n_{k_1} n_{k_2} n_k + n_{k_1} n_{k_2} n_{k_3} - n_{k_1} n_{k_3} n_k - n_{k_2} n_{k_3} n_k] \sim n^3
\]

- Two constant of motion: particle number and energy \Rightarrow Two fluxes

\[
Q \equiv \dot{N} = \int d^3k \, \dot{n}(k) \\
\frac{d}{dt} = \int d^3k \, |k| \, \dot{n}(k)
\]
Kolmogorov-Zakharov (KZ) Spectra

- From collision integral: nonlinear 4-wave interactions

\[ P \sim Q \sim \dot{n} \sim n^3 \quad \Rightarrow \quad n \sim P^{1/3} \sim Q^{1/3} \]

- Dimensional analysis determines uniquely the out-of-equilibrium steady state (KZ) power spectra if the interactions are local in momentum space

\[ n(k) \sim \frac{Q^{1/3}}{k^{4/3}} \quad \text{particle cascade} \]

\[ n(k) \sim \frac{P^{1/3}}{k^{5/3}} \quad \text{energy cascade} \]

- Same exponents for scalar theories in the absence of condensation
Dual cascade: Fjørthoft argument

- Direction of fluxes: Injection of energy at $k_f$ and dissipating at $k_- \ll k_f \ll k_+$

- Direct energy cascade: If energy was dissipating at low momenta then particles would dissipate at a faster rate than the pumping rate!

$$Q_- \sim \frac{P}{k_-} \gg \frac{P}{k_f} \sim Q$$

Fjørthoft (1953)
Are KZ spectra in QCD physically relevant?
Small angle approximation

- Coulomb interaction is singular at small momentum transfer $k \gg q \geq m$

$|\mathcal{M}_{k1\rightarrow 23}|^2 \sim \alpha^2 \frac{s^2}{s^2}$

- Fokker-Plank equation: Diffusion and drag

$\frac{\partial}{\partial t} n_k \equiv \frac{\hat{q}}{4k^2} \frac{\partial}{\partial k} k^2 \left[ \frac{\partial}{\partial k} n_k + \frac{n_k^2}{T_*} \right]$

L. D. Landau (1937) B. Svetitski (1988)

- KZ spectra are not stationary solutions of the collision integral (contrary to non-relativistic Coulomb scattering! A. V. Kats, V. M. Kontorovich, S. S. Moiseev, and V. E. Novikov (1975) )

- $\hat{q}$ diverges in the IR for $n \sim k^{-5/3}$ and for $n \sim k^{-4/3}$ in the UV
Steady state solutions

\[ \frac{\partial}{\partial t} n_k \equiv \hat{q} \frac{\partial}{\partial k} k^2 \left[ \frac{\partial}{\partial k} n_k + \frac{n_k^2}{T_*} \right] + F - D \]

- Thermal fixed point: \[ \frac{T_*}{k - \mu} \]
- Non-thermal fixed point (inverse particle cascade): \[ n(k) \sim \frac{A}{k} > \frac{T_*}{k} \]
  \[ A \equiv \frac{1}{2} T_* \left( 1 + \sqrt{1 + \frac{16Q}{\hat{q}T_*}} \right) \]
- Warm cascade behavior:

  

- No (homogeneous) steady state solution for the energy cascade
Numerical simulation of FK equation with forcing

- The occupation number (left) and, the energy and particle number fluxes (right) at late times
Inelastic processes in the small angle approximation
Effective 3 waves interactions (1 to 2 scatterings)

- **LPM regime**: many scatterings can cause a gluon to branch with the rate

  \[ k \frac{d\Gamma}{dk} \sim \frac{\alpha}{t_f(k)} \sim \alpha \sqrt{\frac{q}{k}} \]

  formation time: \( t_f(k) \sim \frac{k}{k^2} \sim \frac{k}{q t_f} \)

- **Bethe-Heitler regime** for \( t_f(k) < \ell_{mfp} \sim m^2/q \)

  \[ k \frac{d\Gamma}{dk} \sim \frac{\alpha}{\ell_{mfp}} \]


Effective 3 waves interaction (1 to 2 scatterings)

\[
\frac{\partial}{\partial t} n_k \equiv \frac{1}{k^3} \left[ \int_0^\infty dq K(k + q, q) F(k + q, q) - \int_0^k dq K(k, q) F(k, q) \right]
\]

\[
F(k, q) \equiv n_{k+q} n_k + (n_{k+q} - n_k) n_q \sim n^2
\]

\[
K(k, q) \equiv \alpha \sqrt{q} \frac{(k + q)^{7/2}}{k^{1/2} q^{3/2}}
\]

- Direct energy cascade (if interactions are local!)

\[
n_k \sim \frac{p^{1/2}}{\hat{q}^{1/4} k^{7/4}}
\]


Locality of interactions

- Assume a power spectrum $n \sim k^{-x}$ and require the energy flux to be independent of $k$.

- We obtain $x = 7/4$ and

$$P = \alpha \sqrt{q} \int_0^1 dz \frac{(1-z)^x + z^x - 1}{z^{x+1/2}(1-z)^{x+3/2}} \ln \frac{1}{z}$$

- The above integral diverges: $P = \infty$

$\Rightarrow$ Effective 3 waves Interaction is nonlocal in momentum space and the KZ spectrum cannot be realized.
Gradient expansion of the collision integral \(( k \ll k_f )\)

- The collision integral is dominated by strongly asymmetric branchings

- In the regime: \( k \ll k_f \) To the left of the source

\[
\frac{\partial}{\partial t} n_k \approx \frac{1}{k^3} \left[ \int_0^\infty dq K(k+q, q) F(k+q, q) \right] a \approx \alpha \frac{\sqrt{q}}{k^{7/2}} [T_* - k n(k)]
\]

- Late times (steady state) solution is thermal (no fluxes):

\[
n(k) \equiv \frac{T_*}{k}
\]

Gradient expansion of the collision integral \((k \gg k_f)\)

- In the regime: \(k \gg k_f\) To the right of the source. We perform a gradient expansion around \(k \gg q\)

- We obtain an isotropic diffusion equation in “4-D”

\[
\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{\text{inel}}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)
\]

- with the inelastic diffusion coefficient (in the LPM regime)

\[
\hat{q}_{\text{inel}} = \alpha \sqrt{q} \int_0^\infty dq \sqrt{q} n(q)
\]
Gradient expansion of the collision integral ( II )

\[
\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{\text{inel}}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)
\]

- Recall that 3-D diffusion conserves number of particles: 
  \[ N \sim \int dk \, k^2 \, n(k) \]
  Its fixed point (inverse particle cascade):
  \[ n(k) \sim \frac{1}{k} \]

- 4-D diffusion conserves energy: 
  \[ E \sim \int dk \, k^3 \, n(k) \]
  Its fixed point (direct energy cascade):
  \[ n(k) \sim \frac{1}{k^2} \]
Numerical simulation with forcing (full solution)

Parametric estimate:
\[ \hat{q} \sim k_f^3 n^2 \]

\[ n(k) \sim \frac{P}{\hat{q}k^2} \sim \frac{P^{1/3} k_f^{1/3}}{k^2} \]

Nonlocal turbulent spectrum:
hard gluons in the inertial range interact dominantly with gluons at the forcing scale (energy gain)

\[ k_{\text{max}}(t) \sim \sqrt{q_{\text{inel}} t} \]
Interplay between elastic and inelastic processes (1)

- For a spectrum falling faster than $1/k$ one can neglect the drag term in the elastic part. Then, the collision integral in the UV reads

$$\frac{\partial}{\partial t} n(k) \simeq \frac{q_{\text{inel}}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k) + \frac{q_{\text{el}}}{4k^2} \frac{\partial}{\partial k} k^2 \frac{\partial}{\partial k} n(k)$$

$$\equiv \frac{B}{4k^{3-\beta}} \frac{\partial}{\partial k} k^{3-\beta} \frac{\partial}{\partial k} n(k)$$

where $B = q_{\text{inel}} + q_{\text{el}}$ and $\beta = \frac{2}{1 + q_{\text{inel}}/q_{\text{el}}}$

- Steady state solution:

$$n(k) \sim \frac{1}{k^{2-\beta}} \quad 0 < \beta < 1$$
Interplay between elastic and inelastic processes (II)

Exponent $\beta \approx 0.24$ is computed self-consistently

- Elastic processes reduce slightly the exponent at asymptotically late times. At intermediate times $k^{-2}$ spectrum is observed: balance between the drag and diffusion terms?
Wave turbulence in QCD is different from scalar theories. It is characterized by nonlocal interactions in momentum space: Kolmogorov-Zakharov spectra are not physically relevant.

Inelastic processes dominates the dynamics with a direct energy cascade.

To the right of the forcing scale: Kinetic theory predicts a steady state spectrum $\sim k^{-2}$ (in the LPM and BH regimes) in agreement with Hard Loop simulations.

To the left of the forcing scale the system appears to be thermalized: warm cascade.

Outlook: mass corrections, anisotropic fluxes, strong turbulence in the presence of strong fields (on the lattice): different exponents?