Simulating Quantum Fluids

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Why simulate (nearly perfect) quantum fluids?

Hard computational problem: Have to determine real time correlation functions. Achieve quantum supremacy?

Fluid dynamics is the universal effective description of non-equilibrium many body systems. Description is “most effective” in nearly perfect fluids.

Fluid-gravity correspondence: Can (strongly coupled) fluids teach us something about quantum gravity?
Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

\[ \mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 \]

\[ \frac{\partial f_p}{\partial t} + \bar{v} \cdot \nabla_x f_p = C[f_p] \quad \omega < T \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta} \]
Effective theories (Strong coupling)

\[ \mathcal{L} = \bar{\lambda}(i \sigma \cdot D) \lambda - \frac{1}{4} G^a_{\mu \nu} G^a_{\mu \nu} + \ldots \iff S = \frac{1}{2\kappa^2} \int d^5 x \sqrt{-g} R + \ldots \]

\[ SO(d + 2, 2) \to Schr^2_d \quad AdS_{d+3} \to Schr^2_d \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T) \]
Outline

I. EFT: Gradient expansion
II. EFT: Fluctuations
III. Models of fluids: Kinetic theory & QFT
IV. Models of fluids: Holography
V. Analyzing fluids: How to measure $\eta/s$
I. Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{j}^\rho = 0
\]

\[
\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{j}^\epsilon = 0
\]

\[
\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0
\]

Ward identity: mass current = momentum density

\[
\vec{j}^\rho \equiv \rho \vec{v} = \vec{\pi}
\]

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

\[
\Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)
\]
Conformal fluid dynamics: Symmetries

Symmetries of a conformal non-relativistic fluid

Galilean boost \( \vec{x}' = \vec{x} + \vec{v}t \quad t' = t \)

Scale trafo \( \vec{x}' = e^s \vec{x} \quad t' = e^{2s} t \)

Conformal trafo \( \vec{x}' = \vec{x}/(1 + ct) \quad 1/t' = 1/t + c \)

This is known as the Schrödinger algebra (= the symmetries of the free Schrödinger equation)

Generators: Mass, momentum, angular momentum

\[
M = \int dx \rho \quad P_i = \int dx j_i \quad J_{ij} = \int dx \epsilon_{ijk} x_j j_k
\]

Boost, dilations, special conformal

\[
K_i = \int dx x_i \rho \quad D = \int dx x \cdot j \quad C = \int dx x^2 \rho/2
\]
Spurion method: Local symmetries

Diffeomorphism invariance $\delta x_i = \xi_i(x, t)$

$$\delta g_{ij} = -\mathcal{L}_\xi g_{ij} = -\xi^k \partial_k g_{ij} + \ldots$$

Gauge invariance $\delta \psi = i\alpha(x, t)\psi$

$$\delta A_0 = -\dot{\alpha} - \xi^k \partial_k A_0 - A_k \dot{\xi}^k$$
$$\delta A_i = -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + m g_{ik} \dot{\xi}^k$$

Conformal transformations $\delta t = \beta(t)$

$$\delta O = -\beta \dot{O} - \frac{1}{2} \Delta_O \dot{\beta} O$$

More recent work: Newton-Cartan geometry
Example: Stress tensor

Determine transformation properties of fluid dynamic variables

$$\delta \rho = -\mathcal{L}_\xi \rho \quad \delta s = -\mathcal{L}_\xi s \quad \delta v = -\mathcal{L}_\xi v + \dot{\xi}$$

Stress tensor: Ideal fluid dynamics

$$\Pi_{ij}^0 = P g_{ij} + \rho v_i v_j,$$

First order viscous hydrodynamics

$$\delta^{(1)} \Pi_{ij} = -\eta \sigma_{ij} - \zeta g_{ij} \langle \sigma \rangle \quad \zeta = 0$$

$$\sigma_{ij} = \left( \nabla_i v_j + \nabla_j v_i - \frac{2}{3} g_{ij} \langle \sigma \rangle \right) \quad \langle \sigma \rangle = \nabla \cdot v + \frac{\dot{g}}{2g}$$

Son (2007)
Simple application: Kubo formula

Consider background metric $g_{ij}(t, x) = \delta_{ij} + h_{ij}(t, x)$. Linear response

$$\delta \Pi^{xy} = -\frac{1}{2} G^{xyxy}_R h_{xy}$$

Harmonic perturbation $h_{xy} = h_0 e^{-i\omega t}$

$$G^{xyxy}_R = P - i\eta \omega + \ldots$$

Kubo relation:

$$\eta = -\lim_{\omega \to 0} \left[ \frac{1}{\omega} \text{Im} G^{xyxy}_R(\omega, 0) \right]$$

Gradient expansion:

$$\omega \leq \frac{P}{\eta} \simeq \frac{s}{\eta} T.$$
Second order conformal hydrodynamics

Second order gradient corrections to stress tensor

\[
\delta^{(2)} \Pi^{ij} = \eta \tau_\pi \left[ \langle D \sigma^{ij} \rangle + \frac{2}{3} \sigma^{ij} (\nabla \cdot v) \right] \\
+ \lambda_1 \sigma^{(i} \sigma^{j)}^k + \lambda_2 \sigma^{(i} \Omega^{j)}^k + \lambda_3 \Omega^{(i} \Omega^{j)}^k + O(\nabla^2 T)
\]

\[
D = \partial_0 + v \cdot \nabla \quad A^{ij} = \frac{1}{2} \left( A_{ij} + A_{ji} - \frac{2}{3} g_{ij} A^k_k \right) \quad \Omega^{ij} = (\nabla_i v_j - \nabla_j v_i)
\]

New transport coefficients \( \tau_\pi, \lambda_i, \gamma_i \)

Can be written as a relaxation equation for \( \pi^{ij} \equiv \delta \Pi^{ij} \)

\[
\pi^{ij} = -\eta \sigma^{ij} - \tau_\pi \left[ \langle D \pi^{ij} \rangle + \frac{5}{3} (\nabla \cdot v) \pi^{ij} \right] + \ldots
\]

Chao, Schaefer (2011)
“Speed” of diffusive wave in Navier-Stokes theory

$$v_D = \frac{\partial |\omega|}{\partial k} = \frac{2\eta}{\rho} k$$

May encounter $v_D \gg c_s$

Not a fundamental problem (should impose $k < \Lambda$), but a nuisance in simulations.

Second order fluid dynamics, relaxation type

$$i\omega = \frac{\nu k^2}{1 - i\omega\tau_\pi} \quad \text{("resummed hydro")}$$

Limiting speed $v_D^\infty \sim \sqrt{\eta/(\rho\tau_\pi)}$

Find $v_D^\infty \sim c_s$ for $\tau_\pi = \eta/P$. 
II. Beyond gradients: Hydrodynamic fluctuations

Hydrodynamic variables fluctuate

\[ \langle \delta v_i(x, t) \delta v_j(x', t) \rangle = \frac{T}{\rho} \delta_{ij} \delta(x - x') \]

Linearized hydrodynamics propagates fluctuations as shear or sound

\[ \langle \delta v^T_i \delta v^T_j \rangle_{\omega, k} = \frac{2T}{\rho} \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right) \frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \quad \text{shear} \]

\[ \langle \delta v^L_i \delta v^L_j \rangle_{\omega, k} = \frac{2T}{\rho} \hat{k}_i \hat{k}_j \frac{\omega k^2 \Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\omega k^2 \Gamma)^2} \quad \text{sound} \]

\[ \nu = \nu_T + \nu_L: \quad \nabla \cdot \nu_T = 0, \quad \nabla \times \nu_L = 0 \]

\[ \nu = \eta/\rho, \quad \Gamma = \frac{4}{3} \nu + \ldots \]
Hydro Loops: “Breakdown” of second order hydro

Correlation function in hydrodynamics

\[ G_{xyxy}^S = \langle \{ \Pi_{xy}, \Pi_{xy} \} \rangle_{\omega,k} \simeq \rho_0^2 \langle \{ v_x v_y, v_x v_y \} \rangle_{\omega,k} \]

Match to response function in \( \omega \to 0 \) (Kubo) limit

\[ G_{xyxy}^R = P + \delta P - i\omega[\eta + \delta \eta] + \omega^2 [\eta \tau_\pi + \delta(\eta \tau_\pi)] \]

with

\[ \delta P \sim T\Lambda^3 \quad \delta \eta \sim \frac{T\rho\Lambda}{\eta} \quad \delta(\eta \tau_\pi) \sim \frac{1}{\sqrt{\omega}} \frac{T\rho^{3/2}}{\eta^{3/2}} \]
Hydro Loops: RG and “breakdown” of 2nd order hydro

Cutoff dependence can be absorbed into bare parameters. Non-analytic terms are cutoff independent.

Fluid dynamics is a “renormalizable” effective theory.

Small $\eta$ enhances fluctuation corrections: $\delta\eta \sim T \left( \frac{\rho}{\eta} \right)^2 \left( \frac{P}{\rho} \right)^{1/2}$

Small $\eta$ leads to large $\delta\eta$: There must be a bound on $\eta/n$.

Relaxation time diverges: $\delta(\eta\tau) \sim \frac{1}{\sqrt{\omega}} \left( \frac{\rho}{\eta} \right)^{3/2}$

2nd order hydro without fluctuations inconsistent.
Fluctuation induced bound on $\eta/s$

$$(\eta/s)_{min} \simeq 0.2$$

Schaefer, Chafin (2012), see also Kovtun, Moore, Romatschke (2011)
III. Kinetic theory

Microscopic picture: Quasi-particle distribution function $f_p(x, t)$

\[ \rho(x, t) = \int d\Gamma_p \sqrt{gm} f_p(x, t) \quad \pi_i(x, t) = \int d\Gamma_p \sqrt{gp_i} f_p(x, t) \]

\[ \Pi_{ij}(x, t) = \int d\Gamma_p \sqrt{gp_i v_j} f_p(x, t) \]

Boltzmann equation

\[
\left( \frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - \left( g^{il} \dot{g}_{lj} p^j + \Gamma_{jk}^i \frac{p^j p^k}{m} \right) \frac{\partial}{\partial p^i} \right) f_p(t, x, ) = C[f]
\]

\[ C[f] = \]

Solve order-by-order in Knudsen number $Kn = l_{mfp}/L$
Kinetic theory: Knudsen expansion

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \ldots$

Gradient exp. $\delta f_n = O(\nabla^n)$
≡ Knudsen exp. $\delta f_n = O(Kn^n)$

First order result

$\delta^{(1)} \Pi^{ij} = -\eta \sigma^{ij}$

$\eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2}$

Second order result

$\delta^{(2)} \Pi^{ij} = \frac{\eta^2}{P} \left[ \langle D\sigma^{ij} \rangle + \frac{2}{3} \sigma^{ij} (\nabla \cdot v) \right]$

$+ \frac{\eta^2}{P} \left[ \frac{15}{14} \sigma^{ik} \sigma^{jk} - \sigma^{ik} \Omega^{jk} \right] + O(\kappa \eta \nabla^i \nabla^j T)$

relaxation time $\tau_\pi = \eta/P$
Frequency dependence, breakdown of kinetic theory

Consider harmonic perturbation $h_{xy} e^{-i\omega t + ikx}$. Use schematic collision term $C[f_p^0 + \delta f_p] = -\delta f_p / \tau$.

$$
\delta f_p(\omega, k) = \frac{1}{2T} \frac{-i\omega p_x v_y}{-i\omega + i\vec{v} \cdot \vec{k} + \tau_0^{-1}} f_p^0 h_{xy}.
$$

Leads to Lorentzian line shape of transport peak

$$
\eta(\omega) = \frac{\eta(0)}{1 + \omega^2 \tau_0^2}
$$

Pole at $\omega = i\tau_0^{-1}$ ($\tau_0 = \eta/(sT)$) controls range of convergence of gradient expansion.

High frequency behavior misses short range correlations for $\omega > T$. 
Bulk viscosity and conformal symmetry breaking

Conformal symmetry breaking (thermodynamics)

\[
1 - \frac{2\mathcal{E}}{3P} = \frac{\langle \mathcal{O}_c \rangle}{12\pi maP} \sim \frac{1}{6\pi} n\lambda^3 \frac{\lambda}{a}
\]

How does this translate into \( \zeta \neq 0 \)? Momentum dependent \( m^*(p) \).

\[
\text{Im} \Sigma(k) \sim zT \sqrt{\frac{T}{\epsilon_k}} \text{Erf} \left( \sqrt{\frac{\epsilon_k}{T}} \right) \ll T
\]

\[
\text{Re} \Sigma(k) \sim zT \frac{\lambda}{a} \sqrt{\frac{T}{\epsilon_k}} F_D \left( \sqrt{\frac{\epsilon_k}{T}} \right)
\]

Bulk viscosity

\[
\zeta = \frac{1}{24\sqrt{2\pi}} \lambda^{-3} \left( \frac{z\lambda}{a} \right)^2
\]

\[
\zeta \sim \left( 1 - \frac{2\mathcal{E}}{3P} \right)^2 \eta
\]
IIIb. Quantum Field Theory

The diagrammatic content of the Boltzmann equation is known: Kubo formula with “Maki-Thompson” + “Azlamov-Larkin” + “Self-energy”

\[
\begin{array}{c}
\text{classical } 2.77 T^{3/2} \\
\text{viscosity } \eta(\omega=0) \\
\end{array}
\]

Can be used to extrapolate Boltzmann result to \( T \sim T_F \)

Enss, Zwerger (2011), see also Levin (2014)
**Short time behavior: OPE**

Operator product expansion (OPE)

\[ \eta(\omega) = \sum_n \frac{\langle O_n \rangle}{\omega(\Delta_n - d)/2} \quad O_n(\lambda^2 t, \lambda x) = \lambda^{-\Delta_n} O_n(t, x) \]

Leading operator: Contact density (Tan)

\[ O_C = C_0^2 \psi \psi \psi \psi^\dagger = \Phi \Phi^\dagger \quad \Delta_C = 4 \]

\[ \eta(\omega) \sim \langle O_C \rangle / \sqrt{\omega}. \text{ Asymptotic behavior + analyticity gives sum rule} \]

\[ \frac{1}{\pi} \int dw \left[ \eta(\omega) - \frac{\langle O_C \rangle}{15\pi^{3/2} \sqrt{m\omega}} \right] = \frac{\mathcal{E}}{3} \]

IV. Holography

DLCQ idea: Light cone compactification of relativistic theory in $d+2$

$$p_\mu p^\mu = 2p_+ p_- - p^2_- = 0 \quad p_- = \frac{p_{\perp}^2}{2p_+} \quad p_+ = \frac{2n + 1}{L}$$

Galilean invariant theory in $d+1$ dimensions.

String theory embedding: Null Melvin Twist

$$AdS_{d+3} \xrightarrow{\text{NMT}} Schr^2_d$$

$$Iso(AdS_{d+3}) = SO(d+2, 2) \supset Schr(d)$$

Son (2008), Balasubramanian et al. (2008)

Other ideas: Horava-Lifshitz (Karch, 2013)
Schrödinger Metric

Coordinates \((u, v, \vec{x}, r)\), periodic in \(v\), \(\vec{x} = (x, y)\)

\[
ds^2 = \frac{r^2}{k(r)^{2/3}} \left\{ \left[ \frac{1 - f(r)}{4\beta^2} - r^2 f(r) \right] du^2 + \frac{\beta^2 r^4}{r^4} dv^2 - [1 + f(r)] du dv \right\}
\]

\[+ k(r)^{1/3} \left\{ r^2 d\vec{x}^2 + \frac{dr^2}{r^2 f(r)} \right\}\]

Fluctuations \(\delta g^y_x = e^{-i\omega u} \chi(\omega, r)\) satisfy \((u = (r+/r)^2)\)

\[
\chi''(\omega, u) - \frac{1 + u^2}{f(u)u} \chi'(\omega, u) + \frac{u}{f(u)^2} \omega^2 \chi(\omega, u) = 0
\]

Retarded correlation function

\[
G_R(\omega) = \frac{\beta r^3}{4\pi G_5} \frac{\Delta v}{f(u) \chi'(\omega, u)} \frac{\chi(\omega, u)}{u \chi(\omega, u)} \bigg|_{u \to 0}.
\]

Adams et al. (2008), Herzog et al. (2008)
Spectral function

\[ \eta(\omega)/s \]

\[ \omega/(\pi T) \]

\[ \eta(0)/s = 1/(4\pi) \]

\[ \eta(\omega \to \infty) \sim \omega^{1/3} \]

Kubo relation (incl. \( \tau_\pi \)):

\[ G_R(\omega) = P - i\eta \omega + \tau_\pi \eta \omega^2 + \kappa_R k^2 \]

\[ \tau_\pi T = -\frac{\log(2)}{2\pi} \]

\[ AdS_5 : \tau_\pi T = \frac{2 - \log(2)}{2\pi} \]

Range of validity of fluid dynamics: \( \omega < T \)

\[ Sch_2 : \text{Cannot be matched to relaxation type hydro?} \]

Quasi-normal modes

$\text{Sch}_2^2 \quad \text{AdS}_5$

QNM's are stable, $\text{Im } \lambda < 0$.

Pole at $\omega \sim iT$ limits convergence of fluid dynamics.

Modes overdamped in $\text{Sch}_2^2$.

V. Experiments: Elliptic flow

Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy.
Determination of $\eta(n, T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n, T)$. But:

The whole cloud is not a fluid. Can we ignore this issue?

No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.
Possible Solutions

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom \((\mathcal{E}_a; \ a = x, y, z)\)

\[
\frac{\partial \mathcal{E}_a}{\partial t} + \nabla \cdot \vec{j}_a = -\frac{\Delta P_a}{2\tau} \\
\Delta P_a = P_a - P
\]

\[
\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{j}^e = 0 \\
\mathcal{E} = \sum_a \mathcal{E}_a
\]

\(\tau\) small: Fast relaxation to Navier-Stokes with \(\tau = \eta/P\)

\(\tau\) large: Additional conservation laws. Ballistic expansion.
Consider $\eta = \alpha n$ and $\alpha \in [0, \infty)$

Navier-Stokes: Ideal hydro $\rightarrow$ very viscous hydro.

A-hydro: Ideal hydro $\rightarrow$ ballistic expansion.
Anisotropic Hydrodynamics

\[ \eta = \alpha_n n \]

\[ \eta = \alpha_T (mT')^{3/2} \]

\[ \Pi_{xx} (\text{Navier-Stokes}) \]

\[ \Pi_{xx} (\text{A-Hydro}) \]

AVH1 hydro code, M. Bluhm & T.S. (2015)
Outlook

Fluid dynamics as an E(F)T: Many interesting questions remain.

Experiment: Main issue is temperature, density dependence of $\eta/s$. How to unfold?

Need hydro codes that exit “gracefully” (anisotropic hydro, hydro+cascade, or LBE).

Quasi-particles vs quasi-normal modes (kinetics vs holography) unresolved. Need better holographic models, improved lattice calculations.