Thermodynamic properties and transport coefficients of a Fermi gas around unitarity

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Gabriel Wlazłowski (Seattle and Warsaw)
Superconductivity and superfluidity in Fermi systems

- Dilute atomic Fermi gases: \( T_c \approx 10^{-9} \text{ eV} \)
- Liquid \(^3\text{He}\): \( T_c \approx 10^{-7} \text{ eV} \)
- Metals, composite materials: \( T_c \approx 10^{-3} - 10^{-2} \text{ eV} \)
- Nuclei, neutron stars: \( T_c \approx 10^5 - 10^6 \text{ eV} \)
- QCD color superconductivity: \( T_c \approx 10^7 - 10^8 \text{ eV} \)

units \( 1 \text{ eV} \approx 10^4 \text{ K} \)
Why would one want to study cold Fermi gases?

One reason:

(for the nerds, I mean the hard-core theorists, not for the phenomenologists)

Bertsch’s Many-Body X challenge, Seattle, 1999

What are the ground state properties of the many-body system composed of spin $\frac{1}{2}$ fermions interacting via a zero-range, infinite scattering-length contact interaction?
Besides pure theoretical curiosity, this problem is relevant to neutron stars! ... and a few other systems!

What are the scattering length and the effective range?

\[ k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r_{\text{eff}} k^2 + \cdots \]

\[ \sigma = \frac{4\pi}{k^2} \sin^2 \delta_0 + \cdots = 4\pi \frac{a^2}{1 + k^2 a^2} + \cdots \xrightarrow{k \to 0} 4\pi a^2 + \cdots \]

If the energy is small only the s-wave is relevant.
Let me consider at first as an instructive example the hydrogen atom.

The ground state energy could only be a function of:

✓ Electron charge
✓ Electron mass
✓ Planck’s constant

and then trivial dimensional arguments lead to

\[
\begin{align*}
E_{gs} &= \frac{e^4 m}{\hbar^2} \times \frac{1}{2}
\end{align*}
\]

Only the factor \( \frac{1}{2} \) requires some hard work (Quantum Mechanics).
Let me now turn to dilute fermion matter

The ground state energy is given by such a function:

\[
E_{gs} = f(N, V, \hbar, m, a, r_0)
\]

Taking the scattering length to infinity and the range of the interaction to zero, we are left with:

\[
E_{gs} = F(N, V, \hbar, m) = \frac{3}{5} \varepsilon_F N \times \xi
\]

\[
\frac{N}{V} = \frac{k_F^3}{3\pi^2}, \quad \varepsilon_F = \frac{\hbar^2 k_F^2}{2m}
\]
What George Bertsch essentially asked in 1999 is:

\[
\text{What is the value of } \xi \text{?! Is it positive?}
\]

But he wished to know the properties of the system as well:

\[
\text{The system turned out to be superfluid!}
\]

\[
E_{gs} = \frac{3}{5} \varepsilon_F N \times \xi \quad \Delta = \varepsilon_F \times \zeta
\]

\[
\xi = 0.372(5), \quad \zeta = 0.45(5)
\]

Now these results are a bit unexpected.
✓ The energy looks almost like that of a non-interacting system!
(there are no other dimensional parameters in the problem)
✓ The system has a huge pairing gap!
✓ This system is a very strongly interacting one, since the elementary cross section is huge!
The initial Bertsch’s Many Body challenge has evolved over time and became the problem of *Fermions in the Unitary Regime.* (this is part of the BCS-BEC crossover problem)

**In cold old gases one can control the strength of the interaction!**

The system is very dilute, but strongly interacting!

\[
\begin{align*}
n r_0^3 & \ll 1 \\
n |a|^3 & \gg 1
\end{align*}
\]

\[
r_0 \ll n^{-1/3} \approx \lambda_F / 2 \ll |a|
\]
Finite Temperatures

Grand Canonical Path-Integral Monte Carlo

\[ \hat{H} = \hat{T} + \hat{V} = \int d^3x \left[ \psi_\uparrow^\dagger(\vec{x}) \left( -\frac{\hbar^2 \Delta}{2m} \right) \psi_\uparrow(\vec{x}) + \psi_\downarrow^\dagger(\vec{x}) \left( -\frac{\hbar^2 \Delta}{2m} \right) \psi_\downarrow(\vec{x}) \right] - g \int d^3x \ \hat{n}_\uparrow(\vec{x}) \hat{n}_\downarrow(\vec{x}) \]

\[ \hat{N} = \int d^3x \left[ \hat{n}_\uparrow(\vec{x}) + \hat{n}_\downarrow(\vec{x}) \right], \quad \hat{n}_s(\vec{x}) = \psi_s^\dagger(\vec{x}) \psi_s(\vec{x}), \quad s = \uparrow, \downarrow \]

**Trotter expansion**

\[ Z(\beta) = \text{Tr} \ \exp \left[ -\beta \left( \hat{H} - \mu \hat{N} \right) \right] = \text{Tr} \ \left\{ \exp \left[ -\tau \left( \hat{H} - \mu \hat{N} \right) \right] \right\}^{N_\tau}, \quad \beta = \frac{1}{T} = N_\tau \tau \]

\[ E(T) = \frac{1}{Z(T)} \text{Tr} \ \hat{H} \ \exp \left[ -\beta \left( \hat{H} - \mu \hat{N} \right) \right] \]

\[ N(T) = \frac{1}{Z(T)} \text{Tr} \ \hat{N} \ \exp \left[ -\beta \left( \hat{H} - \mu \hat{N} \right) \right] \]

No approximations so far, except for the fact that the interaction is not well defined!
Recast the propagator at each time slice and put the system on a 3D-spatial lattice, in a cubic box of side L=Ns, with periodic boundary conditions.

\[
\exp\left[-\tau\left(\hat{H} - \mu \hat{N}\right)\right] \approx \exp\left[-\tau\left(\hat{T} - \mu \hat{N}\right) / 2\right] \exp(-\tau \hat{V}) \exp\left[-\tau\left(\hat{T} - \mu \hat{N}\right) / 2\right] + O(\tau^3)
\]

**Discrete Hubbard-Stratonovich transformation**

\[
\exp(-\tau \hat{V}) = \prod_{\vec{x}} \sum_{\sigma_{\pm}(\vec{x})=\pm1} \frac{1}{2} \left[ 1 + \sigma_{\pm}(\vec{x}) \hat{n}_{\uparrow}(\vec{x}) \right] \left[ 1 + \sigma_{\pm}(\vec{x}) \hat{n}_{\downarrow}(\vec{x}) \right], \quad A = \sqrt{\exp(\tau g) - 1}
\]

**σ-fields fluctuate both in space and imaginary time**

\[
\frac{m}{4\pi\hbar^2 a} = -\frac{1}{g} + \frac{mk_c}{2\pi^2\hbar^2}, \quad k_c < \frac{\pi}{l}, \quad r_{\text{eff}} = \frac{4}{\pi k_c}
\]

Running coupling constant g defined by lattice
Momentum space

\[ \varepsilon_F, \Delta, T \ll \frac{\hbar^2 \pi^2}{2ml^2} \]

\[ \delta \varepsilon > \frac{2\hbar^2 \pi^2}{mL^2} \]

\[ \varepsilon_F, \Delta \gg \frac{2\hbar^2 \pi^2}{mL^2} \]

\[ \xi_{coh} \ll L = N_s l \]

\[ \delta p > \frac{2\pi \hbar}{L} \]
How to choose the lattice spacing and the box size?

\[ n(k) \]

\[ \frac{2\pi}{L} \]

L – box size

\[ k_{\text{max}} = \frac{\pi}{l} \]

l – lattice spacing
\[ Z(T) = \int \prod_{\vec{x}, \tau} D\sigma(\vec{x}, \tau) \text{Tr} \hat{U}(\{\sigma\}) \]

\[ \hat{U}(\{\sigma\}) = T_{\tau} \prod_{\tau} \exp\{-\tau[\hat{h}(\{\sigma\}) - \mu]\} \]

\[ E(T) = \int \frac{\prod_{\vec{x}, \tau} D\sigma(\vec{x}, \tau) \text{Tr} \hat{U}(\{\sigma\})}{Z(T)} \frac{\text{Tr}[\hat{H}\hat{U}(\{\sigma\})]}{\text{Tr} \hat{U}(\{\sigma\})} \]

\[ \text{Tr} \hat{U}(\{\sigma\}) = \{\det[1 + \hat{U}(\{\sigma\})]\}^2 = \exp[-S(\{\sigma\})] > 0 \]

\[ n_{\uparrow}(\vec{x}, \vec{y}) = n_{\downarrow}(\vec{x}, \vec{y}) = \sum_{k, l < k_c} \varphi_{\vec{k}}(\vec{x}) \left[ \frac{\hat{U}(\{\sigma\})}{1 + \hat{U}(\{\sigma\})} \right]_{\vec{k} \, \vec{l}} \varphi_{\vec{l}}^*(\vec{y}), \quad \varphi_{\vec{k}}(\vec{x}) = \frac{\exp(i\vec{k} \cdot \vec{x})}{\sqrt{V}} \]

\textbf{No sign problem!}

\textbf{One-body evolution operator in imaginary time}

\textbf{All traces can be expressed through these single-particle density matrices}
One can thus determine as a function of $T$, $V$ and chemical potential:

- Total Energy
- Particle number
- Entropy of the system
- Pressure
- Spectrum of fermionic elementary excitations (pairing gap, pseudogap, effective mass, self-energy)
- Long range order, condensate fraction (onset of phase transition, critical temperature)
\[
E_{\text{phonons}}(T) = \frac{3}{5} \varepsilon_F N \frac{\sqrt{3} \pi^4}{16 \xi_s^{3/2}} \left( \frac{T}{\varepsilon_F} \right)^4, \quad \xi_s \approx 0.44
\]

\[
E_{\text{quasi-particles}}(T) = \frac{3}{5} \varepsilon_F N \frac{5}{2} \sqrt{\frac{2\pi \Delta^3 T}{\varepsilon_F^4}} \exp \left( -\frac{\Delta}{T} \right)
\]

\[
\Delta = \left( \frac{2}{e} \right)^{7/3} \varepsilon_F \exp \left( \frac{\pi}{2k_F a} \right)
\]

Bulgac, Drut, and Magierski 
Experiment (about 100,000 atoms in a trap):

*Measurement of the Entropy and Critical Temperature of a Strongly Interacting Fermi Gas,*

**Full ab initio theory** (no free parameters)

FIG. 2. (Color online) Energy $E/E_{\text{FG}}$ (red dots), as obtained by Ku et al. [8]. Our AFQMC results extrapolated to infinite volume are shown by open black circles. The results for $N_x = 8$ (open blue squares) were obtained with the DMC algorithm in Ref. [9]. The green square shows the QMC result of Ref. [20] for $\xi$ at $T = 0$. The inset shows the vicinity of the superfluid phase transition at $T_c/\epsilon_F \simeq 0.15$.

FIG. 4. (Color online) Density $n(\mu, T)$ of the UFG (red circles) as obtained by Ku et al. [8], normalized to the density $n_0(\mu, T)$ of a noninteracting Fermi gas. The notation for the AFQMC results is identical to Fig. 2. The diagrammatic MC results of Refs. [21,22] (solid up and down triangles) and the Bold Diagrammatic MC results of Ref. [23] are shown as well (solid squares). The inset shows the vicinity of the superfluid phase transition at $T_c/\epsilon_F \simeq 0.15$. 
Long range order and condensate fraction

\[ g_2(\vec{r}) = \left( \frac{2}{N} \right)^2 \int d^3 \vec{r}_1 \int d^3 \vec{r}_2 \left\langle \psi_\uparrow^\dagger (\vec{r}_1 + \vec{r}) \psi_\uparrow^\dagger (\vec{r}_2 + \vec{r}) \psi_\downarrow (\vec{r}_2) \psi_\downarrow (\vec{r}_2) \right\rangle \]

\[ \alpha = \lim_{r \to \infty} \frac{N}{2} g_2(\vec{r}) - n(\vec{r})^2, \quad n(\vec{r}) = \frac{2}{N} \int d^3 \vec{r}_1 \left\langle \psi_\uparrow^\dagger (\vec{r}_1 + \vec{r}) \psi_\uparrow (\vec{r}_1) \right\rangle \]

Critical temperature for superfluid to normal transition


Hard and soft bosons: Pilati et al. PRL 100, 140405 (2008)
What is happening in spin imbalanced systems?

Induced P-wave superfluidity (*even though fermions interact in s-wave only*)
Two new superfluid phases where before they were not expected

One Bose superfluid coexisting with one P-wave Fermi superfluid
Two coexisting P-wave Fermi superfluids

Going beyond the naïve BCS approximation


Full momentum and frequency dependence of the self-consistent equations (red)
Dimensional arguments and Legendre transform for unitary Fermi gas

\[ P(\mu_\uparrow, \mu_\downarrow) = \mu_\uparrow n_\uparrow + \mu_\downarrow n_\downarrow - \varepsilon(n_\uparrow, n_\downarrow) = \frac{2}{3} \varepsilon(n_\uparrow, n_\downarrow) \]

\[ P(\mu_\uparrow, \mu_\downarrow) = \frac{2}{5} \beta \left[ \mu_\downarrow h \left( \frac{\mu_\downarrow}{\mu_\uparrow} \right) \right]^{5/2}, \quad \beta = \frac{1}{6\pi^2} \left[ \frac{2m}{\hbar^2} \right]^{3/2} \]

\[ \varepsilon(n_\uparrow, n_\downarrow) = \frac{3}{5} \alpha \left[ n_\uparrow g \left( \frac{n_\downarrow}{n_\uparrow} \right) \right]^{5/3}, \quad \alpha = \frac{(6\pi^2)^{2/3}}{2m} \hbar^2 \]

\[ h(y) = \begin{cases} 
1, & y \leq y_0 < 0 \\
\frac{1+y}{(2\xi)^{3/5}}, & y_1 \leq y \leq 1 
\end{cases} \quad g(0) = 1, \quad g(1) = (2\xi)^{3/5} \quad g''(x) \geq 0 \]

\[ g'(0) \leq Y_0, \quad g'(x) \in g(1) \left[ \frac{Y_1}{1+Y_1}, \frac{1}{2} \right] \]
\[ \frac{\mu_\uparrow - \mu_\downarrow}{2} = \Delta \]

Fully polarized

\[ P_{SF} = \frac{4 \beta}{5 \xi^{3/2}} \left( \frac{\mu_\uparrow + \mu_\downarrow}{2} \right)^{5/2} \]

Unpolarized SF

FIG. 2. (Color online) Example of a function \( h(y) \) and the corresponding function \( g(x) \) shown as thick lines. Maxwell’s construction for phase coexistence leads to a linear \( g(x) \) for \( x \in (0.5, 1.0) \), interpolating between the two pure phases shown with lighter lines. This corresponds to the kink and/or first-order phase transition at \( y = y_0 \) in \( h(y) \). Various other sample functions are lightly sketched within the allowed (dotted) triangular region.

From energy of one spin-down particle in a sea of spin-ups
Asymmetric Superfluid Local Density Approximation

\[
\Omega = -\int d^3r \left[ \epsilon(\vec{r}) - \mu \uparrow n(\vec{r}) - \mu \downarrow n(\vec{r}) - V_{ext}(\vec{r}) n(\vec{r}) \right]
\]

\[
\epsilon(\vec{r}) = \frac{\hbar^2}{2m} \left[ \alpha \uparrow(\vec{r}) \tau \uparrow(\vec{r}) + \alpha \downarrow(\vec{r}) \tau \downarrow(\vec{r}) \right] + \frac{3(3\pi^2)^{2/3}}{10m} \left[ n(\vec{r}) + n(\vec{r}) \right]^{5/3} \beta(\vec{r}) + g_{eff}(\vec{r}) |\nu(\vec{r})|^2
\]

\[
n(\vec{r}) = \sum_{E_n < 0} |u(\vec{r})|^2, \quad n(\vec{r}) = \sum_{E_n < 0} |v(\vec{r})|^2, \quad \nu(\vec{r}) = \frac{1}{2} \sum_{E_n} \text{sgn}(E_n) u(\vec{r}) v^*(\vec{r}),
\]

\[
\tau(\vec{r}) = \sum_{E_n < 0} \left| \vec{\nabla} u(\vec{r}) \right|^2, \quad \tau(\vec{r}) = \sum_{E_n < 0} \left| \vec{\nabla} v(\vec{r}) \right|^2,
\]

\[
\alpha \uparrow(\vec{r}) = \alpha \left[ \frac{n(\vec{r})}{n(\vec{r})} \right], \quad \alpha \downarrow(\vec{r}) = \alpha \left[ \frac{n(\vec{r})}{n(\vec{r})} \right], \quad \beta(\vec{r}) = \beta \left[ \frac{n(\vec{r})}{n(\vec{r})} \right] = \beta \left[ \frac{n(\vec{r})}{n(\vec{r})} \right],
\]

\[
\begin{pmatrix}
T \uparrow(\vec{r}) + U \uparrow(\vec{r}) - \mu \uparrow \\
\Delta(\vec{r})
\end{pmatrix}
\begin{pmatrix}
\Delta^*(\vec{r}) \\
-T \downarrow(\vec{r}) - U \downarrow(\vec{r}) + \mu \downarrow
\end{pmatrix}
\begin{pmatrix}
\frac{E_n}{u(\vec{r})} \\
\frac{v(\vec{r})}{\nu(\vec{r})}
\end{pmatrix}
\]
<table>
<thead>
<tr>
<th>((N_a, N_b))</th>
<th>(E_{FNDMC})</th>
<th>(E_{ASLDA})</th>
<th>Error</th>
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<td>6.6 ± 0.01</td>
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<table>
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<th>(E_{ASLDA})</th>
<th>Error</th>
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<td>((15, 14))</td>
<td>69.126 ± 0.31</td>
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</table>
Red line: Larkin-Ovchinnikov phase (unitary Fermi supersolid)

Black line: normal part of the energy density


\[ E(n_a, n_b) = \frac{3}{5} \frac{(6\pi^2)^{2/3}}{2m} \frac{\hbar^2}{\hbar^2} \left[ n_a g \left( \frac{n_b}{n_a} \right) \right]^{5/3} \]
A Unitary Fermi Supersolid: the Larkin-Ovchinnikov phase

Bulgac and Forbes

NB  This is a gas system at the same time!
Observations: Inconclusive

- Need detailed structure or novel signature

MIT Experimental data from Shin et. al (2008)

Courtesy of M.M. Forbes
The temperature dependence of the spectral weight function

Beyond BCS approximation the delta-functions acquire nonzero widths (due to the finite lifetimes of quasiparticle states generated by collisional effects)

From a talk given by Wlazlowski at INT, Seattle, Spring 2011
Matsubara propagator, spectral function and linear response

\[ G(\tilde{p}, \tau) = \frac{1}{Z} \text{Tr} \left\{ \exp \left[ - (\beta - \tau) (H - \mu N) \right] \psi^\dagger(\tilde{p}) \exp \left[ - \tau (H - \mu N) \right] \psi(\tilde{p}) \right\} \]

\[ = - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega A(\omega, \tilde{p}) \frac{\exp(-\omega\tau)}{1 + \exp(-\omega\beta)} \]

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega A(\omega, \tilde{p}) = 1, \]

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega A(\omega, \tilde{p}) \frac{1}{1 + \exp(\omega\beta)} = n(\tilde{p}), \]

\[ A(\omega, \tilde{p}) \geq 0 \]
Response of the two-component Fermi gas in the unitary regime


\[
\chi(p) = -T \frac{d}{dg} \left. \frac{\text{Tr}\{\exp[-\beta(H-\mu N+g\psi(p))]\psi^\dagger(p)\}}{\text{Tr}\{\exp[-\beta(H-\mu N+g\psi(p))]\}} \right|_{g=0} = -\int_0^\beta d\tau G(p,\tau)
\]

One-body temperature (Matsubara) Green's function
Singular value decomposition (SVD) and maximum entropy method (MEM) reconstruction of the spectral function

\[
G(p, \tau) = \frac{1}{Z} \text{Tr} \left\{ \exp \left[ -\left( \beta - \tau \right) (H - \mu N) \right] \psi^\dagger(p) \exp \left[ -\tau \left( H - \mu N \right) \right] \psi(p) \right\} \\
= -\frac{1}{2\pi} \int d\omega A(p, \omega) \frac{\exp(-\omega \tau)}{1 + \exp(-\omega \beta)}
\]
Using photoemission spectroscopy to probe a strongly interacting Fermi gas

\[ E(N) + \hbar \nu = E(N - 1) + E_k + \frac{\hbar^2 k^2}{2m} + \phi \]
\[ G_{\Pi}(\tilde{q}, \tau) = \frac{1}{V} \left\langle \Pi^{(xy)}_{\tilde{q}}(\tau) \Pi^{(xy)}_{-\tilde{q}}(0) \right\rangle \]

\[ \Pi^{(xy)}(\tau) = e^{-\tau(H-\mu N)} \Pi^{(xy)} e^{\tau(H-\mu N)} \]

\[ G_{\Pi}(0, \tau) = \frac{1}{\pi} \int_{0}^{\infty} d\omega \ \eta(\omega) \omega \frac{\cosh[\omega(\tau - \beta / 2)]}{\sinh(\omega\beta / 2)} \]

\[ \int_{0}^{\infty} d\omega \left[ \eta(\omega) - \frac{C}{15\pi \sqrt{\omega}} \right] = \frac{\epsilon}{3} - \frac{C}{12\pi a}, \quad \eta(\omega) \geq 0 \]

\[ n(p) \approx \frac{C}{p^4} \text{ when } p \to \infty \]

\[ E = TS - pV + \mu N \]

\[ S(x, y) = \frac{\xi(x, y) - \zeta(x, y) + \frac{C(x, y)y}{6\pi N k_F}}{N} \]

where \( x = \frac{T}{\epsilon_F} \), \( y = \frac{1}{k_F a} \), \( E = \frac{3}{5} \epsilon_F \xi(x, y) \), \( \mu = \epsilon_F \zeta(x, y) \)
\[ \chi_s = \lim_{p \to 0} \frac{1}{V} \int_0^\beta d\tau \langle s_z(\vec{p},\tau)s_z(-\vec{p},0) \rangle, \quad s_z(\vec{p},\tau) = n_{\uparrow}(\vec{p},\tau) - n_{\downarrow}(\vec{p},\tau) \]

**Spin susceptibility**
\[ \vec{j}_s = \vec{j}_\uparrow - \vec{j}_\downarrow = \sigma_s \vec{F} \]

\[ \Gamma_{sd} = \frac{n}{\sigma_s}, \quad \sigma_s \geq 0 \]

\[ G_s^{jj}(\vec{q},\tau) = \frac{1}{V} \langle [j_{q\uparrow}(\tau) - j_{q\downarrow}(\tau)] [j_{q\uparrow}^z(0) - j_{q\downarrow}^z(0)] \rangle \]

\[ \vec{j}_s = -D_s \nabla (n_\uparrow - n_\downarrow) \]

\[ D_s = \frac{\sigma_s}{\chi_s} \quad \text{(Einstein relation)} \]

\[ D_s \approx v\lambda \sim 1 \quad \text{(kinetic theory)} \]
KSS conjecture

\[ \frac{\eta}{s} \geq \frac{1}{4\pi} \frac{h}{k_B} \]

Minimum defines a "perfect" fluid

Bound has been proposed on the basis of string theory.
Valid for large class of (string) theories
Saturated for the case of strongly coupled theory.

QGP

RHIC

LHC

\[ \frac{\eta}{s} \]

2008 late 2010 early 2011 late 2011 2012 (Post-QM)

arXiv:1210.5778

He near \( \lambda \)-transition

\[ \frac{E}{E_F} \]


String theory

Slide from a talk given by G. Wlazlowski
Shear viscosity of a unitary Fermi gas
(the only complete ab initio calculation in a Fermi system)

FIG. 3: (Color online) The ratio of the shear viscosity to the entropy density $\eta/s$ as a function of dimensionless temperature for $8^3$-lattice (red) squares and $10^3$-lattice (blue) circles. The error bars only presents the stability of the combined (SVD and MEM) analytic continuation procedure with respect to the change of algorithm parameters, and do not include systematic errors of the entropy determination. By (red) dotted line conservative estimation for the upper bound is depicted. Result of the T-matrix theory are plotted by open (purple) circles [15]. In the high and low temperatures regime known asymptotics are depicted: for $T > 0.3\epsilon_F$ by (green) line prediction of the kinetic theory and for $T < 0.2\epsilon_F$ by (brown) line contribution from phonon excitations [13]. By dashed (black) line the KSS bound is plotted.


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\[ \eta = \alpha \hbar n \]

Elliott, Joseph, and Thomas

Joseph, Elliott, and Thomas,
arXiv:1410.4835
High temperature kinetic theory


\[
t = \frac{T}{\varepsilon_F}, \quad x = \frac{1}{k_F a}
\]

\[
z = \frac{n\lambda^3}{2} \quad \text{(fugacity)}
\]

Expansion to order \(O\left(\left(\frac{\lambda}{a}\right)^2\right)\) and \(O\left(\frac{z}{a}\right)\)
References (QMC only):

- The evolution of the shear viscosity away from unitarity
  G. Wlazlowski, W. Quan, and A. Bulgac (to be submitted soon)

- The temperature evolution of the shear viscosity in a unitary Fermi gas,
  G. Wlazlowski, P. Magierski, A. Bulgac, and K.J. Roche,

- Cooper pairing above the critical temperature in a unitary Fermi gas,
  G. Wlazlowski, P. Magierski, J.E. Drut, A. Bulgac, and K.J. Roche,

- Shear Viscosity of a unitary Fermi gas,
  G. Wlazlowski, P. Magierski, and J.E. Drut,

- Equation of state of the unitary Fermi gas: an update on lattice calculations,
  J.E. Drut, T.A. Lahde, G. Wlazlowski, and P. Magierski,

- Onset of a pseudogap regime in ultracold Fermi gases,
  P. Magierski, G. Wlazlowski, and A. Bulgac,

- Momentum distribution and contact of the unitary Fermi gas,
  J.E. Drut, T.A. Lahde, and T. Ten,

- Quantum Monte Carlo study of dilute neutron matter at finite temperatures,
  G. Wlazlowski and P. Magierski,

- Finite-temperature pairing gap of a unitary Fermi gas by quantum Monte Carlo calculations,
  P. Magierski, G. Wlazlowski, A. Bulgac, and J.E. Drut,

- Quantum Monte Carlo simulations of the BCS-BEC crossover at finite temperatures,
  A. Bulgac, J.E. Drut, and P. Magierski,

- Thermodynamics of a trapped unitary Fermi gas,
  A. Bulgac, J.E. Drut, and P. Magierski,

- Spin ½ fermions in the unitary regime: a superfluid of a new type,
  A. Bulgac, J.E. Drut, and P. Magierski,