Crystalline Confinement and fractional fluxes in Abelian Quantum Link and Dimer models

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**Introduction**

- Lattice gauge theories → fundamental contribution towards understanding of strongly correlated systems.
- Most non-perturbative computations done in Euclidean space with Wilson formulation.
- Ultra-cold atoms toolbox → quantum dynamics of gauge theories.
- Questions of real-time evolution and finite baryon density.
- Alternate formulation of gauge theories (Horn, 1981; Orland, Rohrlich, 1990; Chandrasekharan, Wiese, 1997) and QCD with domain wall fermions (Brower, Chandrasekharan, Wiese, 1999) are particularly relevant.
- These realize **continuous** gauge symmetries using **discrete** quantum link variables, having finite dimensional Hilbert space → extension of Wilson formulation of gauge theories.
- Excellent candidate models to be implemented in cold-atom systems.
- Allows construction of very efficient algorithms to study static properties.
Hamiltonian $U(1)$ LGT: Wilson formulation

- **U(1) gauge invariant Hamiltonian:**

\[
H = \frac{g^2}{2} \sum_{x,i} e_{x,i}^2 - \frac{1}{2g^2} \sum (u + u^\dagger)
\]

- **Operators in the Hamiltonian formulation:**

\[
u = \exp(i\varphi); \quad u^\dagger = \exp(-i\varphi); \quad e = -i\partial\varphi;
\]

\[\Rightarrow \text{are operators in the Hamiltonian formulation, operating in an infinite dimensional Hilbert space on a single link}\]

- **U(1) gauge transformations generated by Gauss Law:**

\[
G_x = \sum_i (e_{x,i} - e_{x^{-1},i}); \quad [G_x, H] = 0
\]

\[
V = \prod_x \exp(i\alpha_x G_x); \quad u'_{xy} = V u_{xy} V^\dagger = \exp(i\alpha_x) u_{xy} \exp(-i\alpha_y)
\]

- **Commutation relations realizing gauge invariance:**

\[
[e, u] = u, \quad [e, u^\dagger] = -u^\dagger
\]

\[
[u, u^\dagger] = 0
\]
Hamiltonian $U(1)$ LGT: Quantum Links

- U(1) gauge invariant Hamiltonian:
\[
H = \frac{g^2}{2} \sum_{x,i} E_{x,i}^2 - \frac{1}{2g^2} \sum \left( U - U^\dagger \right)
\]

- $U = S^1 + iS^2 = S^+; \quad U^\dagger = S^1 - iS^2 = S^-; \quad E = S^3$
  \[
  \Rightarrow \text{are operators in the Hamiltonian formulation, operating in a finite dimensional Hilbert space on a single link}
  \]

- U(1) gauge transformations generated by Gauss Law:
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\[
V = \prod_x \exp(i\alpha_x G_x); \quad U'_{xy} = VU_{xy}V^\dagger = \exp(i\alpha_x)U_{xy}\exp(-i\alpha_y)
\]

- Commutation relations realizing gauge invariance:
\[
[E, U] = U, \quad [E, U^\dagger] = -U^\dagger
\]

- $[U, U^\dagger] = 2E$
The (2+1)-d U(1) Quantum Link model

- Simplest Abelian pure gauge model: with spin $S = 1/2$
  $\rightarrow$ 2-dim Hilbert space per link

  $E|\uparrow\rangle = \frac{1}{2} |\uparrow\rangle$; $E|\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle$; $U|\uparrow\rangle = 0$; $U|\downarrow\rangle = |\uparrow\rangle$; $U^\dagger|\uparrow\rangle = |\downarrow\rangle$; $U^\dagger|\downarrow\rangle = 0$

- $E^2$ contributes a constant for $S = 1/2$.

  $$H = -J \sum \left( U_\square + U_\square^\dagger \right) + \lambda \sum \left( U_\square + U_\square^\dagger \right)^2$$

- Plaquettes are flipped only if they have flux in the right order; second term ($= H_\lambda$) counts the number of flippable plaquettes

  $H = 16$
Gauss Law and Charge Sectors

To define the path integral $Z = \text{Tr} \left( \exp(-\beta H) \mathcal{P}_g \right)$, the Gauss Law must be implemented:

$$\sum_i \left( E_{x,i} - E_{x^{-i},i} \right) = Q_x$$

There is zero charge everywhere (charge-0 sector) unless external static charges are placed at vertices.
Symmetry breaking and phase transitions

- Discrete: Rotation by $\pi/2$, Reflection, Charge Conjugation ($C$), Translation ($T = (T_x, T_y)$)

Charge conjugation: $CU = U^\dagger$; $CE = -E$

Symmetry breaking $\rightarrow$ quantum phase transitions.

Continuous: $U(1)$ center symmetries in x- and y-directions
Diagnosis by Exact Diagonalization

- ED on lattices of $4 \times 4, 4 \times 6, 6 \times 6, 6 \times 8, \ldots$ used to study the system. Quite large by ED standards: $6 \times 6$ has $\sim 16$ million states.

- Volume scaling of the lowest energy states:

![Graph 1](image1)

![Graph 2](image2)

- 2-component order parameter $(M_A, M_B)$ to analyze the symmetry breaking patterns

![Pattern](image3)
Phase diagram

Explored with exact diagonalization and a newly developed cluster algorithm using dualization techniques.

An approximate global SO(2) symmetry is emergent at $\lambda_c$. A description in terms of a low-energy effective theory suggests a weak 1st order transition.
OP distributions from Monte-Carlo

(a) $L = 24a, \lambda = -1, T = 0$, (b) $L = 24a, \lambda \sim \lambda_c, T = 0$, (c) $L = 48a, \lambda \sim \lambda_c, T = 0$, (d) $L = 24a, \lambda = 0, T = 0$
EFT description

- Near $\lambda_c$, ED shows (approximate) finite volume rotor spectra behavior: 
  \[ E_m = \frac{m^2 c^2}{2 \rho L_1 L_2}, \quad m \text{ even.} \]

  Emergence of a $SO(2)$ symmetry which is spontaneously broken.

- EFT description around $\lambda_c$ in terms of the unit vector field 
  \[ \vec{e} = (\cos(\varphi), \sin(\varphi)) \]

  representing the direction of $(M_A, M_B)$.

- $(M_A, M_B)$ indistinguishable from $(-M_A, -M_B) \Rightarrow \mathcal{RP}(1)$ model

\[
S[\varphi] = \int d^3x \frac{1}{c} \left[ \frac{\rho}{2} \partial_\mu \varphi \partial_\mu \varphi + \delta \cos^2(2\varphi) + \epsilon \cos^4(2\varphi) \right]
\]

♠ $\delta$ breaks the emergent $SO(2) \rightarrow Z(4)$,
♠ gives small Goldstone boson mass $M_c = 2 \sqrt{2|\delta|/\rho}$
♠ higher order terms give finite string tension at $\lambda_c$
Mean Field Phase Diagram of the EFT

Solid line is 1st order, dotted lines are 2nd order. Would need "fine-tuning" to make the string tension vanish.
Energy density $\langle H_J \rangle$ of two charges $Q = \pm 2$ placed along the axis on $L = 72a$ lattice
Deconfined Crystal?

Universality arguments predict the finite temperature transition to be of BKT type. Systematic investigation underway; hints of a high-temperature phase with broken $T$ symmetry, which gets smoothly restored with increasing temperature.

Order parameter contour plots $(M_A, M_B)$ for $L=24a; \lambda = 0$; (left) $\beta J=1.4$ and (right) $\beta J=0.8$
Selecting charge sectors: Quantum Dimer Models

Choose the sector of the Link model satisfying the (new) Gauss Law:

\[ G_x |\psi\rangle = (-1)^{x_1+x_2} |\psi\rangle \]

Dimer number at a bond can be connected to the electric flux:

\[ E_{xy} = (-1)^{x_1+x_2} (D_{xy} - \frac{1}{2}) \]
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Dimer Model config

negative
positive
background charges
Candidate phases and the big question

(a) Columnar phase (b) Plaquette phase (c) Staggered phase
Symmetries and results with ED

- Translations combined with charge conjugation: $CT_x, CT_y$
- $\pi/4$ rotation $O$ around a lattice point
- Rotation about a plaquette center combined with charge conj $CO'$
- $U(1)^2$ center symmetries

Quantum num. of ground state $(CT_x, CT_y) = (+, +)$

$E_1 = E_2$ and have quantum numbers $(+, -), (-, +)$; $E_3$ has $(+, +)$

For $\lambda \simeq -0.2$, energy gaps behave as $E_{1,2}, E_3 \sim \exp(-\alpha L_1 L_2)$

For $-0.2 \leq \lambda \leq 0.8$, the state $(-, -)$ with energy $E_4 \approx E_3$ state almost degenerates with the $(+, +)$ state.

For $\lambda \geq -0.2$, $E_{1,2} : E_{3,4} : E_{5,6} : E_{7,8} \approx 1 : 4 : 9 : 16$; approx rotor spectrum
EFT considerations

Effective theory to describe the model around $-0.2 \leq \lambda \leq 1.0$

$$\mathcal{L} = \frac{\rho t}{2} \partial_t \varphi \partial_t \varphi + \frac{\rho}{2} \partial_i \varphi \partial_i \varphi + \kappa (\partial_i \partial_i \varphi)^2 + \delta \cos^2 (4\varphi)$$

$$M_{11} = M_A - M_B - M_C + M_D = M_1 \cos \varphi_1,$$

$$M_{22} = M_A + M_B - M_C - M_D = M_1 \sin \varphi_1,$$

$$M_{12} = M_A - M_B - M_C - M_D = M_2 \cos \varphi_2,$$

$$M_{21} = -M_A + M_B - M_C - M_D = M_2 \sin \varphi_2,$$

and $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2 + \frac{\pi}{4})$, where the $M_{ij}$ are the different order parameters to distinguish the different phases

changing the sign of delta $\Rightarrow$ columnar to plaquette phase
Results from QMC

left: L=12a, right: L=48a; top to bottom: $\lambda = 0.5, 0.8, 0.9$
Evidence for columnar phase

Study the angular histogram of the probability density:
\[ \langle \cos(8\varphi) \rangle = \int_{-\pi}^{\pi} d\varphi p(\varphi) \cos(8\varphi) \]

left: Lattice size \( L = 24a, \beta J = 80 \)
the Plaquette phase exists as an interface separating the two columnar phases. Lattice $16 \times 160$, $\beta J = 100$, $\lambda = -0.5(a)$ and $\beta J = 500$, $\lambda = 0.7(b)$
Potential between two static charges $Q = \pm 2$ separated by distance $r$ along the lattice axis for $\beta J = 100$ and $L = 100a$. 
Energy density $-J\langle U_\square + U_\square^\dagger \rangle$ in the presence of two charges $\pm 2$ (separated by $r = 49a$ for $\lambda = -0.2$ and $\beta J = 72$ on $L = 144a$)
Conclusions

- Although quantum simulating QCD is still far away, many of the simpler models have similar physical phenomena. Very useful for insight into the physics of QCD.
- Proposal for the construction of quantum simulators for the quantum link and quantum dimer models have been presented by colleagues from Innsbruck [see arXiv: 1404.5326 (Quantum Spin Ice) and Annals of Physics 351, 634 (2014) (for QLM)]
- Exciting time when theory and experiment meet together for the lattice gauge theories!
- As pointed out earlier, non-Abelian extensions (all the way upto QCD!) exist, which makes the development of this area exciting.
- Interesting challenges coming up next: demonstrate dimensional reduction in gauge theories (connection with spin-liquids).

Thank You for your attention!