Efimov States as Fields on a Fractal

Ehoud Pazy, Physics Department, NRCN

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Seemingly Un-Connected Topics

• Efimov states - Three body universal states recently experimentally observed
• Fields on fractals - essentially a theoretical object of research
Outline

- Introduction
  - Some Efimov state properties
  - Discrete scaling functions
  - Hints on mathematical connections
- Efimov physics in terms of EFT
  - Neumann series solution of integral equation
  - Conditions for discrete scaling solution
- Scaling parameters and conditions on them
- Connecting to one dimensional physics
- Identifying the fractal structure
- Wild speculations
- Summary
Efimov States

In 1970 Vitaly Efimov discovered that in the limit where:
Two-body scattering length diverges $a \rightarrow \pm \infty$
No two body bound state is found but there are an infinite number of three body (trimer) states

“From questionable to pathological to exotic to a hot topic”

*Nature Physics* 5, 533 (2009)
The Experimental Search for Efimov States

- Experiments with $^4\text{He}$ atoms ($T \sim 0.3 \text{ mk}$)
- Nucleons: triton (pnn) and $^3\text{He}$ (ppn)
- Halo Nuclei
- Cold atom physics $^{133}\text{Cs}$ atoms 2005

Kraememr et al (Innsbruck)
Properties of Efimov Trimers

Universal three body physics:
The physics depends only on the scattering length and a further quantity called the “Three body parameter”

Details of the short range interaction become irrelevant

Infinitely many bound states
Properties of Efimov Trimers

Discrete scaling:

- Binding energies: $\lambda^{-2} = e^{-2\pi/s_0}$
- Sizes differ: $\lambda = e^{\pi/s_0}$

The scattering amplitude is a log periodic function:

$$a(p) = A \cos \left( s_0 \ln \frac{p}{\Lambda} + \delta \right)$$
The Efimov Spectrum

\[ E_T^{(n)} \rightarrow \left( e^{-2\pi s_0} \right)^{n-n^*} \frac{\hbar^2 \kappa^2}{m} \]

\[ n \rightarrow \infty \quad a = \pm\infty \]

- Shows a discrete symmetry
- Is infinite
- Universal

Defined by two parameters

The scaling parameter: \( \lambda_0 = e^{\pi/s_0} \)

The three body parameter: \( \kappa^* \)
Quick Introduction to Discrete Scaling

A fractal is an iterative structure
Fractals Can be Very Fancy
Equation for Discrete Symmetry

• Equation for discrete symmetry

\[ f(x) = g(x) + \frac{1}{b} f(ax) \]

where \( f(x) \) is a periodic function

a and b are scaling parameters

\( g(x) \) some initial function

• Its solution is of the form

\[ f(x) = x^{\frac{\ln b}{\ln a}} G \left( \frac{\ln x}{\ln a} \right) \]

where \( G \) is a periodic function
Equation for Discrete Symmetry Iterative Form

• Equation for discrete symmetry

\[ f(x) = g(x) + \frac{1}{b} f(ax) \]

• The equation can be iterated to give

\[ f(x) = \sum_{n=0}^{\infty} b^{-n} g(a^n x) \]
“Spectrum” and the Mellin Transform

Iterative solution:

\[ f(x) = \sum_{n=0}^{\infty} b^{-n} g(a^n x) \]

Mellin transform:

\[ M_f(s) \equiv \int_0^\infty dx \ x^{s-1} f(x) \]

Mellin transform of Iterative solution:

\[ M_f(s) = M_g(s) \frac{ba^s}{ba^s - 1} \]

“Spectrum”:

\[ s_n = \frac{\ln(1/b)}{\ln a} + \frac{2i \pi n}{\ln a} \]
# Hints to the Connection to Quantum Fields on a Fractal

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Efimov States through an Effective Field Theory

- Three body Bosonic Theory
- Effective Field Theory in term of a “di-baryon” two boson field “d”

\[
L = \psi^+ \left( i \partial_0 + \frac{\nabla^2}{2M} \right) \psi + \Delta d^+d - \frac{g}{\sqrt{2}} (d^+ \psi \psi + H.c) + h d^+d \psi^+ \psi
\]

- Dressing of the di-atom propagator “d”

The Integral Equation

\[ a_{ad}(p) = K(p, k) + \frac{2}{\pi} \int_{0}^{\Lambda} dq K(p, q) \frac{q^2}{q^2 - k^2 - i\varepsilon} a(q) \]

\[ K(p, q) = \frac{2}{\sqrt{3}q} \ln \left( \frac{q^2 + pq + p^2}{q^2 - pq + p^2} \right) \]

Solution Asymptotic Homogeneous Case

The asymptotic homogeneous equation:

\[
\text{for } \frac{1}{a_2} \ll p \ll \Lambda \text{ but } k \sim \frac{1}{a_2}
\]

\[
a(p) = \frac{4}{\sqrt{3}\pi} \int_0^\infty \frac{dq}{q} a(q) \ln \left( \frac{q^2 + pq + p^2}{q^2 - pq + p^2} \right)
\]

suggests an ansatz \( a(p) \sim p^s \) this works if \( S \) satisfies

\[
1 - \frac{8}{\sqrt{3}} \frac{\sin \left( \frac{\pi s}{6} \right)}{s \cos \left( \frac{\pi s}{2} \right)} = 0 \quad \Rightarrow \quad S = \pm iS_0 \quad S_0 = 1.0064
\]

Solution with Finite Cut-Off

- Returning to the equation with a finite cut-off, the following solution is obtained:

\[ a(p) = A \cos \left( s_0 \ln \frac{p}{\Lambda} + \delta \right) \]
Making a Connection to Fractals Through the Integral Equation

\[ a(p) = K(p, k) + \frac{2}{\pi} \int_0^{\Lambda} dq K(p, q) \frac{q^2}{q^2 - k^2 - i \epsilon} a(q) \]

Writing out the formal solution in terms of a Neumann series

\[ a(p) = K(p, k) + \sum_{n=1}^{\infty} \lambda^n \psi_n(p) \quad \lambda = \frac{2}{\pi} \]

\[ \psi_n(p) = \int_0^{\Lambda} \ldots \int_0^{\Lambda} K(p, q_1) K(q_1, q_2) \ldots K(q, k) dq_1 \ldots dq_n \]
Discrete Scale Self Similar Solution

- A self similar discrete scaling solution

\[ a(p) = K(p, k) + \sum_{n=1}^{\infty} \lambda^n \psi_n(p) \quad \lambda = \frac{2}{\pi} \]

\[ \psi_n(p) = \int_0^{\Lambda} \cdots \int_0^{\Lambda} K(p, q_1)K(q_1, q_2)\cdots K(q, k) dq_1 \cdots dq_n \]

\[ a(p) = \sum_{n=1}^{\infty} \lambda^n K(\gamma^n p, k) \]

- Can be found if

\[ \int_0^{\Lambda} dqK(p, q)K(q, k) = K(\gamma p, k) \]
The Connection

If

\[ \int_0^\Lambda dq K(p,q)K(q,k) = K(\gamma p, k) \]

Then

\[ a(p) = \sum_{n=1}^{\infty} \lambda^n K(\gamma^n p, k) \]

\[ \gamma \leftrightarrow a \]
\[ \lambda^{-1} \leftrightarrow b \]
Questions

• What are the values of the scaling parameters?
• How to obtain the Efimov spectrum?
• What is the fractal like structure?
Determining the Discrete Scaling Factor

- Performing a Mellin Transformation

\[ M_f(s) \equiv \int_0^\infty dp p^{s-1} f(p) \]

on both sides of

\[ \int_0^\Lambda dk K(q,k)K(k,p) = K(\gamma q, p) \]

The scaling factor \( \gamma \) is determined through

\[ \gamma^{-s} = M_{K \times p}(s) \]

\[ M_{K \times p}(s) = \int_0^\infty dp p^s K(1, p) \]
The Connection Between the Discrete Scaling Parameters

The two expressions are equivalent when

\[ a^{-s} = M_{K \times p}(s) \]

Instead of

\[ s_n = \frac{\ln(1/b)}{\ln a} + \frac{2i\pi n}{\ln a} \]

\[ S = \pm iS_0 \quad S_0 = 1.0064 \]
An Alternative Method to Obtain the Scaling Factor

Discrete scale invariance

\[ a_{sc} (ap) = ba_{sc} (p) \]

Power like solution to homogeneous integral eq.

\[ a_{sc} (p) \sim p^s \]
\[ a^{-s}b = 1 \]

Instead of

\[ S_n = \frac{\ln(1/b)}{\ln a} + \frac{2i \pi n}{\ln a} \]

\[ S = \pm iS_0 \quad S_0 = 1.0064 \]
Viewing Things in a Different Way

- Connecting to Bloch states
- Obtaining Efimov spectrum from Bohr-Sommerfeld quantization.
A Connection to Bloch States

\[ f(p) = p^{-mG} \left( \frac{\ln(p)}{\ln(a)} \right) \]

\[ m = \frac{\ln(b)}{\ln(a)} \]

change coordinates to:

\[ \tilde{\chi} = \ln(p) \]

denoting:

\[ k = im \]

obtain a Bloch function

\[ f(\tilde{\chi}) = e^{ik\tilde{\chi}} G \left( \frac{\tilde{\chi}}{l} \right) \]

lattice constant

\[ l = \ln(a) \]

wave number

\[ k = i \frac{\ln(b)}{\ln(a)} \]
Connecting to Bloch states

Instead of \( s_n = \frac{\ln(1/b)}{\ln a} + \frac{2i \pi n}{\ln a} \) only two values \( s_n = \pm is_0 \)

then we have

\[
is_0 = -\frac{\ln(b)}{\ln(a)} + \frac{2\pi \ln}{\ln(a)}
\]

associating the wave number to \( k = i \frac{\ln(b)}{\ln(a)} \)
we get \( s_0 = k + \frac{2\pi n}{\ln(a)} \)

associating the lattice constant as \( l = \ln(a) \)
so \( s_0 \) can be considered as an effective crystal momentum

\[
s_0 = k + \frac{2\pi n}{l}
\]
Importing Results from Functions with Scaling Symmetry

Since the integral equation is now a scaling symmetry equation it’s solution is given by

\[
f(x) = x \ln \frac{b}{a} G \left( \frac{\ln x}{\ln a} \right) \quad a_{sc}(p) = p^m G \left( \frac{\ln(p)}{\ln(a)} \right) \quad m = -\frac{\ln(\lambda)}{\ln(a)}
\]

since

\[
G \left[ \frac{\ln p}{\ln a} \right] = \sum_{n=-\infty}^{\infty} C_n \exp \left( 2\pi i n \frac{\ln p}{\ln a} \right) \quad \ln a = \frac{2\pi n}{s_0}
\]

\[
a_{sc}(p) \approx A \cos \left( s_0 \ln \frac{p}{\Lambda} + \delta \right)
\]
Spectrum from the Bhor-Sommerfeld Quantization

$S_3$ plays the roll of effective momentum to obtain the spectrum we can use the Bohr-Sommerfeld quantization

$$\int p_i \, dq_i = 2\pi n_i$$

$$s_0 \int d\tilde{x} = n\pi \quad \Rightarrow \quad \ln(p) = \frac{n\pi}{s_0}$$

$$\tilde{x} = \ln(p)$$

$$E_T^{(n)} = -\left(\frac{\pi}{s_0}\right)^n e^{-2\pi/s_0} \frac{\hbar^2 \kappa^2}{m}$$
What is the Fractal?

It is...

Luke Rogers, Prof. of Mathematics
University of Connecticut
A Spiral

\[ a^t \quad t \in \mathbb{R} \]
Transform the plain wave eigenfunctions of the Laplacian

\[ \exp[ikx] \Rightarrow \exp[ikz] \]
The phase of $a$

The scaling parameters: $b$ is real but it seems that $a$ has a phase

Express $a = e^{i \theta_0 + 2\pi n/s_0 + l}$ from $a^{-is_0}b = 1$, $e^{ils_0} = 1$

$$a^{\theta_0s_0}b = 1$$
$$a^{-\theta_0s_0}b = 1$$

$cosh[s_0 \theta_0] = b$
The phase of $a$

The scaling parameters: $b$ is real but it seems $a$ has a phase

$$a^{-s}b = 1 \rightarrow e^{-is_0 \ln(a)} b = 1$$

$$e^{s_0 \text{Im}\{\ln(a)\}} = b$$

$$S = \pm iS_0$$

$$\cosh[s_0 \text{Im}\{\ln(a)\}] = b$$

$$e^{is_0 \text{Re}\{\ln(a)\}} = 1$$

$$\text{Re}\{\ln(a)\} = \frac{2\pi n}{s_0}$$

$S_3$ plays the role of effective momentum
Wild Speculations

Actually the quantization is

\[ s_0 \int_{\tilde{x}_1}^{\tilde{x}} d\tilde{x} = (n + \delta)\pi \]

Where \(\delta\) results from the boundary conditions and determines the three body parameter \(K_*\)

\[ s_0 \int_{\tilde{x}_1}^{\tilde{x}} d\tilde{x} = (n + \delta)\pi \quad \longrightarrow \quad \ln(p) = \frac{n\pi}{s_0} \]

\[ K_* = e^{-\pi\delta/s_0} \ln(\tilde{x}_1) \]

\[ E_T^{(n)} = -\left(e^{-2\pi/s_0}\right)^{n-n^*} \frac{\hbar^2 K_*^2}{m} \]
Wild Speculations

Can the non-trivial phase $\delta - s_0 \int_{\tilde{x}_1}^{\tilde{x}} d\tilde{x} = (n + \delta)\pi$

Be connected top the Zak phase a non-trivial geometric phase obtained in solid.
The phase is called after it’s discoverer Prof. Joshua Zak from the Technion.
In 1989, Professor Zak identified the geometrical phases in the band theory of one-dimensional solids. When a particle travels "slowly" along the energy band and completes a closed loop it acquires a geometrical phase that has significant physical consequences for the properties of materials, which can be determined by the "quantum geometry" of the crystal.
Summary

• Obtained a known result through a different formalism.
• Sheds light on the connection between fields on fractals and the Skorniakov-Ter-Marirosian (SKM) equation.
• Obtained a physical realization of a “quantum field” on a “fractal”.