UNIVERSAL BOUND STATES IN CONFINED GEOMETRIES

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Qu: How does confinement impact few-body bound states?

- Experimentally achievable in the cold-atom system
  - Tunable dimensionality & interactions:
  - Different atomic species (fermions or bosons)

Quasi-2D geometry

Short-range s-wave interactions

\[ V_{\text{lat}} = V_0 \sin^2(k_L) \]

A stack of pancake shaped gases created by imposing a 1D optical lattice on a 3D atomic gas,

\[ V_{\text{lat}} \approx z^2, \text{oscillator potential} \]

Close to the centre of the pancake

Strong harmonic confinement in one direction:

\[ z \approx x = y \]

Tune between 2D and 3D via optical lattice depth,

Dimensionality depends on "F\sim z", "B\sim z", \(k_B T\sim z\)" 

Interactions depend on "B"
OUTLINE

✦ Identical bosons in 3D – The Efimov effect

✦ Evolution towards 2D
  • Quasi-2D geometry

✦ Three-boson problem in quasi-2D
  • Trimer spectra & wave functions
  • Hyperspherical potentials

✦ Two-component quasi-2D Fermi system

✦ Conclusions
Identical bosons in 3D

- V. Efimov (1970): Three identical bosons with resonant short-range interactions ($1/a=0$) support an infinite number of trimers.

- Experimental evidence: Gas of Cs atoms (Kraemer et al., Nature 2006)

- Three-body problem has discrete scaling symmetry:

  - Trimers can be mapped onto another via transformation:
    \[
    E \rightarrow \lambda_0^{-2n} E \quad a \rightarrow \lambda_0^n a
    \]

  - Trimer energies at resonance:
    \[
    -\lambda_0^{-2n} \frac{\hbar^2 \kappa_n^2}{m}
    \]

\[\lambda_0 \simeq 22.7\]
Identical bosons in 2D

- Only two universal trimers:
  
  \[ -16.5 |E_b| \]
  
  \[ -1.27 |E_b| \]

  Bruch & Tjon, PRA 1979

Two-body binding energy

\[ E_b = -\frac{\hbar^2}{ma_{2D}^2} \]

- Dimers and trimers always exist for arbitrarily weak attractive interactions
- Three-body problem exhibits a continuous scaling symmetry
Evolution from 3D to 2D?
**Quasi-2D system**

- **Harmonic confinement along** $z$: $V(z) = \frac{1}{2}m\omega_z^2 z^2$
  - Generated by optical lattice or trap in experiment
  - Bose gas is kinematically 2D when $k_B T \ll \hbar \omega_z$

- **Two-body problem:**

![Graph](image)

$$l_z = \sqrt{\frac{\hbar}{m\omega_z}}$$
Quasi-2D system

- Confinement raises threshold of free atom continuum
- Always have a two-body bound state
- Obtain 2D limit when interactions are weak $|a| \ll l_z$
  - Two-body binding energy $E_b = -\frac{B}{m \pi l_z^2} e^{\sqrt{2\pi l_z/a}}$

Three-body problem characterised by 2 dimensionless parameters:

$|a_-|/a \quad C_z \equiv |a_-|/l_z$

Petrov & Shlyapnikov, PRA 2001
Three bosons in quasi-2D

Hamiltonian:

\[ \hat{H} = \sum_{k,n} (\epsilon_k + n\hbar\omega_z) a_{kn}^{\dagger} a_{kn} \]
\[ + \frac{1}{2} \sum_{k,n_1,n_2,k',n_3,n_4} e^{-k^2/\Lambda^2} e^{-k'^2/\Lambda^2} \langle n_1n_2 | \hat{g} | n_3n_4 \rangle \ a_{q/2+k,n_1}^{\dagger} a_{q/2-k,n_2}^{\dagger} a_{q/2-k',n_3} a_{q/2+k',n_4} \]

UV cut-off fixes \( a_- \) in 3-body problem

Trimer wave function:

\[ \sum_{k_1,k_2} \psi_{k_1,k_2}^{n_1n_2n_3} a_{k_1,n_1}^{\dagger} a_{k_2,n_2}^{\dagger} a_{-k_1-k_2,n_3}^{\dagger} |0\rangle \]
Three bosons in quasi-2D

- Simplifying the calculation:

\[ \chi^{N_2 N_3}_{k_2} = g \sum_{k_1, n_{13}} e^{-k_1^2/\Lambda^2} f_{n_{13}} (N, N_2, n_{13} | n_1 n_2 n_3) \psi^{n_1 n_2 n_3}_{k_1 k_2} \]

\[ \sum_{k_z} e^{-k_z^2/\Lambda^2} \tilde{\phi}_{n_{13}} (k_z) \]

- Depends on only 2 parameters after dropping N (CoM)

- Wavefunction for atom-pair motion:

\[ \psi(\rho, Z) \equiv R^{3/2} \sum_{k, N} e^{i k \cdot \rho} \phi_N(Z) \chi^N_k \]

- Integral equation:

\[ \mathcal{T}^{-1} (k_1, E_3 - \epsilon_{k_1} - N_1 \omega_z) \chi^{N_1}_{k_1} = 2 \sum_{k_2, n_{23} n_{31}} f_{n_{23}} f_{n_{31}} (N_1 n_{23} | N_2 n_{31}) e^{-(k_1^2 + k_2^2)/\Lambda^2} \chi^{N_2}_{k_2} \]

\[ \frac{E_3 - \epsilon_{k_1} - \epsilon_{k_2} - \epsilon_{k_1 + k_2} - (N_1 + n_{23}) \omega_z} {E_3 - \epsilon_{k_1} - \epsilon_{k_2} - \epsilon_{k_1} + k_2} \]

Levinsen, Massignan & MMP, arXiv:1402.1859
Clebsch-Gordon coefficients

- The challenge: to describe the 3D regime with a discrete energy scaling of 515, we require > 515$^3$ coefficients...

$$\langle N_1 n_{23} | N_2 n_{31} \rangle$$

- Use Schwinger’s mapping of the 2D isotropic harmonic oscillator to the SU(2) representation of angular momentum algebra:

$$\mathbf{J} = \frac{1}{2} \left( b_1^\dagger \ b_2^\dagger \right) \sigma \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right)$$

$$|\Psi(j,m)\rangle = |j + m, j - m\rangle$$

$$\langle N_1 n_{23} | N_2 n_{31} \rangle = d\left( \frac{N_1 + n_{23}}{2}, \frac{N_2 - n_{31}}{2}, \frac{N_1 - n_{23}}{2} \right) (2\pi/3)$$

Wigner $d$-matrix: $d_{m'm}^{(j)}(\beta) \equiv \langle \Psi(jm') | e^{-i\beta \mathbf{J}_y} | \Psi(jm) \rangle$
Trimer spectra

- Extra length scale $l_z$ removes weakest bound Efimov states
- Discrete scaling symmetry is only exists for $|a_-| \ll |a| \ll l_z$
- Moderate confinements $C_z \sim 1$:

$^{133}$Cs: $\omega_z \approx 2\pi \times 5\text{kHz}$
$\omega_z \approx 2\pi \times 30\text{kHz}$

- Deepest trimer persists and exists above 3D continuum

$C_z \equiv |a_-|/l_z$
Trimer spectra

$^{133}\text{Cs: } \omega_z \approx 2\pi \times 5\text{kHz}$

- 2D limit recovered for small and negative scattering length
- Two deepest trimers stabilised
- Appearance of 3rd trimer for weaker confinement
- Spectrum exhibits avoided crossings

$C_z \equiv |a_-|/l_z$
Hyperspherical potentials

\[ R^2 = r_1^2 + r_2^2 + r_3^2 \]

- When \( l_z/a < -1 \) potential has repulsive barrier \( \sim 0.15/ma^2 \)
- For \( R \gg l_z \), it resembles 2D hyperspherical potential
- Superposition of 2D and 3D trimers!

Shape of the trimers

2D regime

Levinsen, Massignan & MMP, arXiv:1402.1859
Experimental consequences

- Experiments are often performed at confinements even weaker than 5kHz
- 3D physics will thus impacts three-body correlations in realistic 2D gases when $a < 0$
- Confinement raises continuum by $\hbar \omega_z$
- Trimer resonance & loss peak disappear once $l_z / |a_-| \lesssim 2.5$

**Immediate consequence for the quest to observe discrete scaling symmetry:**
- 2nd trimer signature disappears once $l_z / |a_-| \lesssim 22.7 \times 2.5$, corresponding to $\omega_z \approx 2\pi \times 10\text{Hz}$ in the case of $^{133}\text{Cs}$

2D Bose gas experiments: Paris, Innsbruck, Chicago, Monash…
Two-component Fermi systems

- Quasi-2D formalism can be generalised to more particles and mass-imbalanced fermions
- Existence of *universal* tetramer in 3+1 system – no dependence on UV cut-off

![Graph showing the relationship between mass ratios and energy differences for dimers, trimers, and tetramers.](image)

Levinsen & MMP, PRL 110, 055304 (2013)

3D tetramer: Blume, PRL 2012
Concluding remarks

• Quasi-2D confinement fundamentally impacts Efimov trimers
  • Deepest trimer remains 3D-like even under strong confinement
  • Hybridisation with 2D-like trimers stabilises the two deepest trimers for all negative scattering lengths
    • Use this to engineer more stable Efimov-like hybrid trimers?
  • Confinement is expected to have a similar effect on 4-body, 5-body etc. Efimov states

• Universal tetramers in the two-component Fermi system
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Hyperspherical expansion

✧ Usual expression for the wave function:

\[ \Psi(R, \Omega) = \frac{1}{R^{5/2} \sin(2\alpha_k)} \sum_{n=0}^{\infty} f_n(R) \Phi_n(R, \Omega) \]

✧ Further expand the angular part:

\[ \Phi_n(R, \Omega) = \sum_{m} \eta_{nm}(R, \alpha_k) \ h_m(R, \Omega) \]

\[ h_m = \tau_{m_1}(R \sin \alpha_k, \theta_{ij}) \ \tau_{m_2}(R \cos \alpha_k, \theta_{k,ij}) \]

where \( \tau(X, \theta) \) obeys the equation:

\[ \left( -\frac{l_z^2}{X^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{X^2}{l_z^2 \cos^2 \theta} \right) \tau = 2 \mu \tau \]

- evolves into harmonic oscillator for \( X \gg l_z \)