Super Efimov effect

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**Few-body problems**

They are challenging but useful:

- Newton gravity $\rightarrow$ perturbation theory, chaos
- Quantum atoms $\rightarrow$ variational Hartree-Fock
- Quantum molecules $\rightarrow$ Born-Oppenheimer

*Efimov effect is “new” entry*
Few-body universality

- Low energies, short-range interactions
- Universal regime: $a \gg$ other length scales
- Two-body bound state near resonance

$$E_D = \frac{\hbar^2}{ma^2} \quad \text{for} \quad a > 0$$
Efimov problem

Three bosons near resonance:

Universality

At resonance as $n \rightarrow \infty$

$$\frac{E_T^{(n+1)}}{E_T^{(n)}} \rightarrow e^{-2\pi/s_0}$$
**Basics intuition**

- How can short-range forces create infinite number of bounds states?
- Born-Oppenheimer approximation:

  - Due to separation of scales, long-range effective potential!
  - Window of universality
  - Scale-invariant potential
Energy spectrum

\[ V(R) = -\frac{1/4 + s_0^2}{R^2} \]

\[ E_n = -\frac{s^2}{R_0^2} \exp \left( -\frac{2\pi n}{s} + \theta \right) \]

Landau & Lifshitz: Fall to center for strong attraction

\[ s_0 > 0 \]

Efimov geometric spectrum
Experimental signatures

Three-body loss:

\[ \dot{n} = -L_3 n^3 \]

enhanced when trimers merge with atom threshold

First experiment
Innsbruck 2006
### Beyond standard model?

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<tr>
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<th>$d = 2$</th>
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<tr>
<td>p-wave</td>
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No scale invariant two-body attraction away from 3d s-wave!

Efimov effect was liberated from 3d

*Nishida, Tan*
**Beyond standard model?**

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<td>X</td>
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No Efmov effect, but ...
\textbf{p-wave in 2d}

Square well solution

\[
\frac{dJ_l(kr)}{dr} \frac{J_l(kr)}{dK_l(\kappa r)} = \frac{dK_l(\kappa r)}{dr} 
\]

In p-wave critical attraction needed

\[ V_0 r_0^2 = 5.784 \]

Normalized wave-function

\[
\psi(r) = \frac{\kappa}{\sqrt{2\pi}} \frac{K_1(\kappa r)}{\sqrt{\ln(\kappa r_0)}} 
\]

- No scale invariance
- Point-like boson as \( r_0 \rightarrow 0 \)
This talk

Few-body quantum physics of resonantly interacting fermions in flatland
Superfluids

- T=0 state of a neutral many-body system
- No dissipation, quantum vortices, ...
- Old: \(^4\text{He}\) and \(^3\text{He}\)
- New: Bose and Fermi ultracold atoms
P-wave superfluids

From mean-field: Volovik, Read, Green,...

- Chiral condensate $\Delta_p = (p_x \pm ip_y) \hat{\Delta}$ preferred
- Topological phase transition at $\mu = 0$
- Chiral Majorana modes on boundaries
- Toy model for a film of $^3\text{He}$

Sometimes mean-field is not good enough near resonance!
Super Efimov effect

At resonance near threshold:

- Infinite tower of $l = \pm 1$ trimer bound states
  
  \[
  E_3^{(n)} \propto \exp\left(-2e^{3\pi n/4+\theta}\right)
  \]

- Infinite set of $l = \pm 2$ tetramer resonances
  
  \[
  E_4^{(n)} \propto \exp\left(-2e^{3\pi n/4+\theta-0.188}\right)
  \]

Super exponential scaling!
P-wave model in d=2

\[ \mathcal{L} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2} \right) \psi + \phi_a^\dagger \left( i\partial_t + \frac{\nabla^2}{4} - \varepsilon_0 \right) \phi_a + g \phi_a^\dagger \psi (-i\nabla_a) \psi + g \psi^\dagger (-i\nabla_{-a}) \psi^\dagger \phi_a + \nu_3 \psi^\dagger \phi_a^\dagger \phi_a \psi + \nu_4 \phi_{-a}^\dagger \phi_a \phi_{-a} \phi_a + \nu'_4 \phi_{-a}^\dagger \phi_a \phi_a \phi_a \phi_a \]

spinless \hspace{1cm} \text{composite}

fermion \hspace{1cm} l = \pm 1 \hspace{0.5cm} \text{boson}

- P-wave resonance ↔ zero energy bound state
- All dimensionless couplings are included
Efimov effect from RG

Flow of atom-dimer vertex:

\[ \partial_t \chi = \partial_t \chi + \partial_t \chi + \partial_t \chi + \partial_t \chi \]

**RG=one-loop diagrams**

**bosons vs fermions in 3d**

\[ T \sim \frac{1}{s_0} \]

\[ \lambda_{3R} \]

Tetramers can be found from RG

Limit cycle

Tuesday, April 1, 14
**Super Efimov from RG**

**Two-body:**

Perturbative counting is reliable!

\[ s = \ln \frac{\Lambda}{k} \]

\[ g^2(s) = \frac{1}{s \pi} + \frac{1}{g^2(0)} \]

Irrelevant in IR like QED

**Three-body:**

Double log periodic solution:

\[ v_3(s) \to \frac{2\pi}{s} \left[ 1 - \cot \left( \frac{4}{3} (\ln s - \theta) \right) \right] \]

Divergences= trimer bound states
One-channel model:

\[
H = \int \frac{dk}{(2\pi)^2} \frac{k^2}{2} \psi_k^\dagger \psi_k - \nu_0 \sum_{a=\pm} \int \frac{dk dp dq}{(2\pi)^6} \partial_x \chi_a(p) \partial_x \chi_{-a}(q) \psi_{\frac{k}{2}+p}^\dagger \psi_{\frac{k}{2}}^\dagger \psi_{\frac{k}{2}}^\dagger \psi_{\frac{k}{2}-q} \psi_{\frac{k}{2}+q}^\dagger
\]

Separable interaction

\[
\chi_a(p) = p_a e^{-p^2/(2\Lambda^2)}
\]
T-matrix solution

Two-fermion scattering T-matrix:

\[ T(E; \mathbf{p}, \mathbf{q}) = \frac{16\pi |\mathbf{p}| |\mathbf{q}| \cos(\varphi_p - \varphi_q) e^{-(p^2 + q^2)/(2\Lambda^2)}}{\frac{2\pi}{v_0} - \Lambda^2 - E e^{-E/\Lambda^2} E_1(-E/\Lambda^2)} \]
Three-fermion scattering T-matrix:

Near binding energy \( T_{ab}(E; \vec{p}, \vec{q}) \rightarrow Z_a(\vec{p})Z_b^*(\vec{q})/(E + \kappa^2) \)

\[
Z_a(p) = - \int \frac{d\mathbf{q}}{2\pi} \frac{(p+2q)_a e^{-(5p^2 + 5q^2 + 8p\cdot q)/(8\Lambda^2)}}{p^2 + q^2 + p\cdot q + \kappa^2} \times \frac{\sum_{b=\pm} (2p+q)_b Z_b(q)}{(\frac{3}{4}q^2 + \kappa^2) e^{(\frac{3}{4}q^2 + \kappa^2)/\Lambda^2} E_1((\frac{3}{4}q^2 + \kappa^2)/\Lambda^2)}
\]
Partial wave decomposition:

\[ Z_\alpha(p) = e^{i\ell \varphi} p z_\alpha(p) \]

s and d waves are coupled!

Similar to deuteron due to tensor one-pion force
T-matrix solution

Analytic solution

leading log approximation

Numeric solution

\[ \lambda_n \equiv \ln \ln \Lambda / \kappa_n \]

<table>
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<th>( \lambda_n )</th>
<th>( \lambda_n - \lambda_{n-1} )</th>
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<td>( \infty )</td>
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<td>2.35619</td>
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Near binding energy

\[ T_{ab}(E; \vec{p}, \vec{q}) \to Z_a(\vec{p})Z_b^*(\vec{q})/(E + \kappa^2) \]

\[ E_3^{(n)} \propto \exp(-2e^{3\pi n/4+\theta}) \]

Agreement with RG result
Effective potential

\[ V(R) = -\frac{1}{4R^2} - \frac{1/4 + r^2}{(R \ln \frac{R}{R_0})^2} \]

Semiclassical solution with double Langer correction

\[ E_n = -\frac{r^2}{R_0^2} \exp \left( -2e^{\frac{\pi}{r} n + \theta} \right) e^{2\left(\frac{\pi}{r} n + \theta\right)} \]

super exponential scaling
• Adiabatic approximation

\[ \Psi_{l=1} = R^{-3/2} f_{l=1}(R) \Phi_{l=1}(\Omega; R) \]

• s-d wave mixing is well captured

• Diagonal corrections important for super Efimov effect

• Is adiabatic approximation reliable?
**Tetramer states in 3d**

- **Universal tetramer resonances**

  ![Diagram showing tetramer states in 3d](image)

- **Universality and # tetramers not settled**

  ![Innsbruck text](image)
Tetramers from RG

\[ \frac{dv'}{ds} = -\frac{4g^4}{\pi} + \frac{2g^2v_3}{\pi} - \frac{2g^2v'_4}{\pi} + \frac{2v'_4}{\pi} \]

- Numerical solution necessary
- Singularities understood analytically

\[ E_4^{(n)} \propto \exp\left(-2e^{3\pi n/4+\theta-0.188}\right) \]

Similar to Efimov tetramers:

Tetramer ground state

Tetramer resonances
Super Efimov in 3d?

- Recent RG calculation includes trimer degrees of freedom
- Suggests super Efimov tower of tetramers for every Efimov trimer in 3d!

\[ k_4^{(n)} = k_3 \exp(\alpha e^{-\beta n}) \]

- Hand-waving RG argument: appears due to logarithmic trimer divergences that feed into the four-body solution.
Experiments

- Great success in three dimensions
- Quasi 2d fermions near p-wave resonance
- Trimers sizes:

<table>
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<tr>
<th>n</th>
<th>GS</th>
<th>l</th>
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<th>3</th>
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<tbody>
<tr>
<td>size</td>
<td>Å</td>
<td>μ</td>
<td>$10^{38}$m</td>
<td>$10^{499}$m</td>
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- No tuning possible in this theory!
Open questions

- Decay of tetramers from RG
- Tetramers and higher-body from T-matrix?
- Superfluid near resonance
Conclusion

• Super Efimov -- double exponential scaling
• Many question to be asked and answered...
Ultracold atoms

- Ensembles of neutral alkali atoms
- Low densities $n \sim 10^{14} \text{cm}^{-3}$ → gases
- Laser cooling $T \sim 10^{-9} \text{K}$ → quantum
- Tunable interactions and geometry
- Harmonic trap keeps atoms together
Quantum simulator

- Quantum
- High degree of tuning
- Table-top size

“Let nature do the calculation”

- Lattice models → high Tc superconductors
- Artificial gauge fields → topological states of matter
- Precision measurements → equation of state of neutrons
- Few-body physics → quantum chemistry
- Single atom manipulation → quantum computer
Experimental achievements

BEC@JILA&MIT 1995

Vortices@MIT 2005

Mott shells@ MPI&Harvard 2010

EOS@ENS&MIT 2012
Quantum simulator

• Quantum
• High degree of tuning
• Table-top size

“Let Nature do the calculation”

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• Artificial gauge fields \(\rightarrow\) topological states of matter
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Feshbach resonance

Tunable interactions in ultracold gases

Interaction strength tuned by magnetic field B
Feshbach resonance

Tunable interactions in ultracold gases

Resonance phenomenon:

\[ a(B) \approx a_{bg} \left[ 1 + \frac{\Delta B}{B - B_0} \right] \]