The Coulomb Problem in Momentum Space without Screening

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(The TORUS Collaboration)
Physics Problem:

Nuclear Reactions dominated by few degrees of freedom

Reactions: Elastic Scattering, Breakup & Transfer

$^{3}\text{He}(d,p)^{4}\text{He}$

Light nuclei

$^{140}\text{Sn}(d,p)^{141}\text{Sn}$

Heavy nuclei
Physics Problem:

Nuclear Reactions dominated by few degrees of freedom

Reactions: Elastic Scattering, Breakup & Transfer

Theory: few-body techniques

Continuum Discretized Coupled Channel Faddeev

Three-particles

$^3\text{He}(d,p)^4\text{He}$

$^{140}\text{Sn}(d,p)^{141}\text{Sn}$
Reduce Many-Body to Few-Body Problem

**Task:**
- Isolate important degrees of freedom in a reaction
- Keep track of important channels
- Connect back to the many-body problem

Hamiltonian for effective few-body problem:

\[ H = H_0 + V_{np} + V_{nA} + V_{pA} \]

- np interaction
- Optical potentials p+A and n+A

**Three-Body Problem**
(d,p) Reactions as three-body problem


Applied Faddeev AGS Equations to $^{12}\text{C}(d,p)^{13}\text{C}$

Elastic, breakup, rearrangement channels are included and fully coupled
(compared to e.g. CDCC calculations)
(d,p) Reactions as three-body problem

Elastic, breakup, rearrangement channels are included and fully coupled (compared to e.g. CDCC calculations)

**Issue:** current momentum space implementation of Coulomb interaction (screening) does not converge for $Z \geq 20$

Courtesy: F.M. Nunes
Solve Faddeev equations in Coulomb basis (no screening)

Scattering: Faddeev equations best solved in momentum space

Partial-wave Coulomb function in momentum space

Very nasty! Oscillatory singular at \( p=q \)

Matrix elements in Coulomb basis:

Example: plane wave basis: \( V(p',p) \equiv <p' | V | p> \)
Coulomb basis: 2 singularities, for \( p'=p \): “pinch” singularity

Up to now

- not directly solved
- Indirect: Chinn, CE, Thaler, PRC44, 1569 (1991) for p+A scattering

Work with separable functions:

\[ V(p',p) \equiv \sum g(p') \lambda g(p) \]

Can we handle this?
First Test in Two-Body System

Separable t-matrix derived from p+A optical potential (generalized EST scheme)

\[ t_l(E) = \sum_{i,j} u_i f_{l,k_{Ei}} \tau_{ij}(E) f_{l,k_{Ej}}^* u_j \]

Nuclear matrix elements \( \langle p | t_l(E) | p' \rangle \)

\[ \langle p | u_i f_{l,k_E} \rangle = t_l(p, k_E; E_{k_E}) \equiv u_l(p) \]
\[ \langle f_{l,k_E}^* | u | p' \rangle = t_l(p', k_E; E_{k_E}) \equiv u_l(p') \]

Coulomb distorted nuclear matrix element

\[ \langle \psi_{i,p}^C | u_i f_{l,k_E} \rangle = \int_0^\infty \frac{dq}{2\pi^2} \frac{q^2}{2} u_l(q) \psi_{i,p}^C(q) \equiv u_l^C(p) \]
\[ \langle f_{l,k_E}^* | u_{i,p}^C \rangle = \int_0^\infty \frac{dq}{2\pi^2} \frac{q^2}{2} u_l(q) \psi_{i,p}^C(q) \equiv u_l^C(p)^\dagger \]

\( \psi_{p\alpha l}^C(p) \) is the Coulomb scattering wave function
Challenge I: momentum space Coulomb functions

General: \[ \psi_{q,\eta}^{C(+)}(p) = \lim_{\gamma \to +0} \int d^3r \ e^{-ipr-\gamma r} \psi_{q,\eta}^{C(+)}(r) \]
\[ = -4\pi e^{-\pi \eta/2} \Gamma(1 + i\eta) \lim_{\gamma \to +0} \frac{d}{d\gamma} \left\{ \frac{[p^2 - (q + i\gamma)^2]^{i\eta}}{[|p - q|^2 + \gamma^2]^{1+i\eta}} \right\} \]

Partial wave decomposition (Mukhamedzhanov, Dolinskii) (1966)

\[ \psi_{q,\eta}^{C(+)}(p) \equiv \sum_{l=0}^{\infty} (2l + 1) \psi_{l,q,\eta}^{C}(p) P_l(\hat{p} \cdot \hat{q}) \quad \psi_{l,q,\eta}^{C}(p) = \frac{1}{2} \int_{-1}^{1} dz \ P_l(z) \psi_{q,\eta}^{C(+)}(p). \]

\[ \frac{1}{2} \int_{-1}^{1} dz \ P_l(z)(\zeta - z)^{-1-i\eta} = \frac{e^{\pi \eta}}{\Gamma(1 + i\eta)} (\zeta^2 - 1)^{-i\eta/2} Q_l^{i\eta}(\zeta) \]

Essential: \[ Q_l^{i\eta}(\zeta) \] has different representations depending on \( \zeta \)
$Q_l^{i\eta}(\zeta)$ has different representations in terms of the hypergeometric function $_2F_1(a;b;c;z)$ depending on $\zeta$

$\zeta$ large enough ( $p$ and $q$ different)  \quad \Rightarrow \quad \text{“regular” representation}

$$Q_l^{i\eta}(\zeta) = \frac{e^{-\pi \eta} \Gamma(l + i\eta + 1) \Gamma(1/2)}{2^{l+1} \Gamma(l + 3/2)} (\zeta^2 - 1)^{i\eta/2} \zeta^{-l - i\eta - 1}$$

$$\times \ _2F_1 \left( \frac{l + i\eta + 2}{2}, \frac{l + i\eta + 1}{2}; l + \frac{3}{2}; \frac{1}{\zeta^2} \right)$$

$\zeta \approx 1$ ( $p \approx q$ ) \quad \Rightarrow \quad \text{“pole-proximity” representation}

$$Q_l^{i\eta}(\zeta) = \frac{1}{2} e^{-\pi \eta} \left\{ \Gamma(i\eta) \left( \frac{\zeta + 1}{\zeta - 1} \right)^{i\eta/2} \ _2F_1 \left( -l, l + 1; 1 - i\eta; \frac{1 - \zeta}{2} \right)$$

$$+ \ \frac{\Gamma(-i\eta) \Gamma(l + i\eta + 1)}{\Gamma(l - i\eta + 1)} \left( \frac{\zeta - 1}{\zeta + 1} \right)^{i\eta/2} \ _2F_1 \left( -l, l + 1; 1 + i\eta; \frac{1 - \zeta}{2} \right) \right\}$$
Partial-wave momentum space Coulomb functions

“regular” representation

$$\psi_{i,q}^C(p) = -\frac{4\pi\eta e^{-\pi\eta/2} q(pq)^l}{(p^2 + q^2)^{1+l+i\eta}} \left[ \frac{\Gamma(1 + l + i\eta)}{(1/2)_{l+1}} \right]$$

$$\times \frac{1}{2} F_1 \left( \frac{2 + l + i\eta}{2}, \frac{1 + l + i\eta}{2}; l + 3/2; \frac{4q^2p^2}{(p^2 + q^2)^2} \right)$$

$$\times \lim_{\gamma \to 0} \left[ p^2 - (q + i\gamma)^2 \right]^{-1+i\eta}$$

“pole-proximity” representation:

$$\psi_{i,q}^C(p) = -\frac{2\pi}{p} \exp(-\pi\eta/2 + i\sigma_l) \left[ \frac{(p + q)^2}{4pq} \right]^l \lim_{\gamma \to 0} 2 \Im D.$$
$q = 1.5 \text{ fm}^{-1}$
Work in progress: publish code in CPC
**Challenge II:**
Matrix elements with Coulomb basis functions

Separable t-matrix derived from p+A optical potential (generalized EST scheme)

\[ t_l(E) = \sum_{i,j} u|f_{l,k_{E_i}}\rangle \tau_{ij}(E) \langle f^*_{l,k_{E_j}}|u \]

Nuclear matrix elements \( \langle p|t_l(E)|p'\rangle \)

\[
\langle p|u|f_{l,k_{E}} \rangle = t_l(p,k_E;E_{k_E}) \equiv u_l(p) \\
\langle f^*_{l,k_{E}}|u|p' \rangle = t_l(p',k_E;E_{k_E}) \equiv u_l(p')
\]

Coulomb distorted nuclear matrix element

\[
\langle \psi^C_{l,p}|u|f_{l,k_{E}} \rangle = \int_0^{\infty} \frac{dq \ q^2}{2\pi^2} \ u_l(q)\psi^C_{l,p}(q)^* \equiv u^C_l(p) \\
\langle f^*_{l,k_{E}}|u|\psi^C_{l,p} \rangle = \int_0^{\infty} \frac{dq \ q^2}{2\pi^2} \ u_l(q) \ \psi^C_{l,p}(q) \equiv u^C_l(p)^\dagger
\]

“oscillatory” singularity at \( q = p \):

\[
\lim_{\gamma \to 0} \frac{1}{(q - p + i\gamma)^{1+\eta}}
\]
Gel’fand-Shilov Regularization:
Generalization of Principal value regularization
Idea: reduce value of integrand near singularity

\[
\int_{-\Delta}^{\Delta} dx \frac{\varphi(x)}{x^{1+\iota \eta}} = \int_{-\Delta}^{\Delta} dx \frac{\varphi(x) - \varphi(0) - \varphi'(0)x}{x^{1+\iota \eta}}
\]

simplified

- \frac{i\varphi(0)}{\eta} \left[ \Delta^{-\iota \eta} - (\Delta)^{-\iota \eta} \right] + \ldots

➢ Reduce integrand around pole by subtracting 2 terms of the Laurent series
Gel’fand-Shilov Regularization:
Generalization of Principal value regularization
Idea: reduce value of integrand near singularity

\[
\int_{-\Delta}^{\Delta} dx \frac{\varphi(x)}{x^{1+i\eta}} = \int_{-\Delta}^{\Delta} dx \frac{\varphi(x) - \varphi(0) - \varphi'(0)x}{x^{1+i\eta}} - \frac{i\varphi(0)}{\eta} [\Delta^{-i\eta} - (\Delta)^{-i\eta}] + \ldots
\]

- Reduce integrand around pole by subtracting 2 terms of the Laurent series

no Coulomb

\( Re u_1 \) [fm^2] vs. \( p \) [fm\(^{-1}\)]

- (a) \( p + ^{12}\text{C} \)
- (b) \( p + ^{48}\text{Ca} \)
- (c) \( p + ^{208}\text{Pb} \)

- (d) \( 1 = 0 \)
- (e) \( 1 = 4 \)
- (f) \( 1 = 8 \)
\[ p + ^{12}\text{C} \]

\[ u_l^C(p_\alpha) = \int_0^{p_\alpha - \Delta} \frac{dp}{2\pi^2} \frac{p^2 u_l(p) \psi_{p_\alpha l}(p)}{2}\]

\[ + \int_{p_\alpha - \Delta}^{p_\alpha + \Delta} \psi_{p_\alpha l}(p) + \int_{p_\alpha + \Delta}^{\infty} \psi_{p_\alpha l}(p) \]

Graph showing the real part of the wave function for \( C\) nuclei with different \( \Delta \) values.
First Physics Check: \( p + ^{48}\text{Ca} \)

Selected partial wave S-matrix elements \( S_{l+1} \) for \( p + ^{48}\text{Ca} \) (CH89 optical potential) with Coulomb distorted \( n + ^{48}\text{Ca} \) formfactors

Method not designed for two-body scattering!
Summary & Outlook

Faddeev-AGS framework in Coulomb basis passed first test!

- Momentum space nuclear form factors obtained in a Coulomb distorted basis for high charges for the first time.
- “Oscillatory singularity” of $\psi_{q,l}^c(p)$ at $p \rightarrow q$ successfully regularized.
- Algorithms to compute $\psi_{q,l}^c(p)$ and overlap integrals successfully implemented.

Near Future:

Implementation of Faddeev-AGS equations in Coulomb basis
TORUS: Theory of Reactions for Unstable Isotopes

A Topical Collaboration for Nuclear Theory

http://www.reactiontheory.org/

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