AT THE THRESHOLD OF THE EFIMOV EFFECT

AU SEUIL DE L’EFFET EFIMOV

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OUTLINE OF THE TALK

- Basic facts and physical motivation
- The physical model and the integral equation to solve
- Useful limiting cases, their solution
- The analytical results
- A numerical study
BASIC FACTS AND PHYSICAL MOTIVATION
THE EFIMOV EFFECT

Relevant regime:

- a resonant $s$-wave binary interaction between particles
- assume infinite scattering length, no two-body bound state

Then the Efimov effect may occur:

- an infinite number of trimer states
- the spectrum is asymptotic to a geometric sequence, in the limit of a large quantum number $n$:

  $$ E_n \xrightarrow{n \to +\infty} E_{\text{glob}} e^{-2\pi n/|s|} $$

- the exponent $s \in i\mathbb{R}^+$ is given by Efimov zero-range theory, contrarily to $E_{\text{glob}}$
- spectrum becomes geometric, as in zero-range theory, when de Broglie wavelength $\gg$ interaction ranges
A PARTICULARLY INTERESTING CASE

There exists a control parameter $\alpha$ allowing one to continuously switch on/off the Efimov effect.

How does the system evolve from a finite number to an infinite number of trimer states?

Simple facts:

- The efimovian states cannot emerge from $E = -\infty$ [any physical spectrum is bounded from below], they shall emerge from $E = 0$.
- Close to threshold, the efimovian states are in the zero-range regime so their spectrum shall be entirely geometric.
• behavior of exponent $s$ known, vanishes as $(\alpha - \alpha_c)^{1/2}$:
  \[ \Lambda(s, \alpha) = 0 \]
  with $\Lambda$ even function of $s$. At threshold, collision in $s = 0$ of two real ($\alpha < \alpha_c$) or imaginary ($\alpha > \alpha_c$) roots:
  \[ \frac{1}{2}s^2 \partial_s^2 \Lambda(0, \alpha_c) + (\alpha - \alpha_c) \partial_{\alpha} \Lambda(0, \alpha_c) = O((\alpha - \alpha_c)^2) \]

• Does $E_{\text{glob}}$ also vanish or diverge at the threshold, with some critical exponent?

Our goal here:

• Answer this question quantitatively on a simple but realistic model: the infinitely narrow Feshbach resonance

• Then analytic techniques exist to calculate $E_{\text{glob}}$, as done for three bosons (Gogolin, Mora, Egger, 2008).

• Also three-body losses suppressed in that limit
THE PHYSICAL MODEL AND
THE INTEGRAL EQUATION TO SOLVE
Configuration & Predictions of Efimov Theory

Make Efimov effect avoidable thanks to Pauli exclusion principle:

- polarized fermions do not interact in $s$-wave
- so take two same-spin-state fermions of mass $m$ resonantly interacting ($1/a = 0$) with an impurity of mass $M$
- Control parameter is mass ratio $\alpha = m/M$: no Efimov effect if $\alpha$ not too large (Efimov, 1973)

Even more interesting: a sequence of efimovian thresholds

- in the sectors of increasing odd angular momenta:
  \[
  \alpha_c^{(l=1)} = 13.60696 \ldots \quad \alpha_c^{(l=3)} = 75.99449 \ldots \\
  \alpha_c^{(l=5)} = 187.9583 \ldots \quad \alpha_c^{(l=7)} = 349.6384 \ldots 
  \]
- no Efimov effect for even angular momenta
Simple Born-Oppenheimer explanation:

\[ \psi(r_1, r_2, R) \approx \phi(r_1 - r_2) \Phi(R; r_1, r_2) \]

where the ground-state wavefunction \( \Phi \) of the impurity at fixed fermionic positions is a symmetric function of these positions, of eigenenergy \(-\hbar^2C^2/(2Mr_{12}^2)\) [here \( C = \exp(-C) \)], which leads to the effective potential

\[ V_{\text{eff}}(r_{12}) = \frac{\hbar^2l(l + 1)}{mr_{12}^2} - \frac{\hbar^2C^2}{2Mr_{12}^2} \equiv \frac{\hbar^2(s_l^2 - 1/4)}{mr_{12}^2} \]

We have results beyond the Born-Oppenheimer approximation:

At bounded distance from threshold:

\[ |s_l|^2 \xrightarrow{l \to \infty} \frac{1}{2}C^2(\alpha - \alpha_c^{(l)})[1 + O(1/l)] \]

\[ \frac{1}{2}C^2\alpha_c^{(l)} \xrightarrow{l \to \infty} \left( l + \frac{1}{2} \right)^2 + \frac{17 - C^2}{12} - \frac{7}{6} \left( C + \frac{1}{C + 1} \right) + O(1/l) \]
WHICH IMPURITY-FERMION INTERACTION

- A Feshbach resonance: two-channel model
- in the open channel, van de Waals interaction of length $b$ and non-resonant scattering length $a_{bg} \approx b$
- infinitely narrow: take limit $b \to 0$ for fixed (rather than diverging) interchannel coupling $\Lambda$. Then corresponding Feshbach length $R_*$ does not vanish. E. g. for $|a_{bg}| \ll b$:

$$R_* \simeq \frac{\pi \hbar^4}{\Lambda^2 \mu^2}$$

- $R_*$ gives the effective range of the binary interaction:

$$f_k = \frac{-1}{ik + k^2 R_*}$$
Ansatz for the trimer state of energy \( E = -\hbar^2 q^2 / (2\mu) < 0 \):

\[
|\psi_{3 \text{ at}}\rangle = \int \frac{\prod_{i=1}^{3} d^3 k_i}{[2\pi]^3} (2\pi)^3 \delta \left( \sum_{i=1}^{3} k_i \right) A(k_1, k_2, k_3) a^\dagger_{k_1} c^\dagger_{k_2} c^\dagger_{k_3} |0\rangle
\]

\[
|\psi_{1 \text{ at }+1 \text{ mol}}\rangle = \int \frac{d^3 k}{(2\pi)^3} B(k) b^\dagger_{-k} c^\dagger_k |0\rangle
\]

- Integral equation from Schrödinger’s equation:

\[
\left[ q_{\text{rel}}(k) + q_{\text{rel}}^2(k) R_* \right] D(k) = - \int \frac{d^3 k'}{2\pi^2} \frac{D(k')}{q^2 + k^2 + k'^2 + \frac{2\alpha}{1+\alpha} k \cdot k'}
\]

where \( D(k) \simeq B(k) \) for \( |a_{bg}| \ll b \)

- Effective relative wavenumber between impurity and fermion:

\[
q_{\text{rel}}(k) = \left[ q^2 + \frac{1 + 2\alpha}{(1 + \alpha)^2} k^2 \right]^{1/2}
\]

- At fixed angular momentum: \( D(k) = d(k) Y_i^0(\hat{k}) \)
USEFUL LIMITING CASES, THEIR SOLUTION
We shall obtain the trimer energies analytically with a relative error $O(q R_*)$ by matching two solutions:

When $q R_* \ll 1$ there exists a momentum interval where both solutions are applicable and are in their $k \to \infty$ and $k \to 0$ asymptotic regimes. Matchable asymptotic forms:

$$k^2 d(k) \xrightarrow{k/q \to \infty} e^{i\theta}(k/q)^s + c.c. + O(k/q)^2$$

$$k^2 d(k) \xrightarrow{k R_* \to 0} e^{i\theta}(k R_*)^s + c.c. + O(k R_*)$$
HOW TO SOLVE?

$E < 0, R_* = 0$:

- Fourier transform the real space Efimov solution

$E = 0, R_* > 0$ (Gogolin, Mora, Egger, 2008):

- integral term is scaling invariant. Change of variable $x = \ln(kR_* \cos \nu)$ [where $\nu = \arcsin \frac{\alpha}{1+\alpha}$ is mass angle] makes it translationally invariant: setting $k^2 d(k) = F(x)$,

$$0 = (1 + e^x)F(x) + (K \ast F)(x)$$

- Fourier transform with respect to $x$:

$$0 = \tilde{F}(S+i) + \Lambda_l(iS, \alpha) \tilde{F}(S)$$

- Infinite product representation of $s \mapsto \Lambda_l(s)$ over its roots and poles. Then solution for $\tilde{F}(S)$ is an infinite product of ratios of $\Gamma$ functions $[\Gamma(z+1) = z\Gamma(z)]$
THE ANALYTICAL RESULTS
Exact value of the global energy scale:

\[ E_{\text{glob}}^{(l)} = -\frac{2\hbar^2}{\mu R_*^2} e^{2\theta_l/|s_l|} \equiv -\frac{\hbar^2 q_{\text{glob}}^{(l)^2}}{2\mu} \]

\[ \theta_l = \text{Im}[\ln \Gamma(1+s_l) + \ln \Gamma(1+2s_l) + 2 \ln \Gamma(l+1-s_l) + \ln \Gamma(l+2-s_l)] \]

\[ + \int_{|s_l|}^{s_l} dS \ln \left[ \Lambda_l(iS, \alpha) \frac{S^2 + (l + 1)^2}{S^2 - |s_l|^2} \right] \]

\[ + \sum_{k \geq 1} \frac{(-1)^k B_{2k}}{(2k)!} \frac{d^{2k-1}}{dS^{2k-1}} \left\{ \ln \left[ \Lambda_l(iS, \alpha) \frac{S^2 + (l + 1)^2}{S^2 - S_l^2} \right] \right\}_{S=|s_l|} \]

Finite limit at threshold:

\[ \theta_l/|s_l| \rightarrow 3\psi(1) - 2\psi(l + 1) - \psi(l + 2) + \sum_j \left[ \psi(x_j) + \psi(1 + x_j) - \psi(l + 1 + 2j) - \psi(l + 2 + 2j) \right] \]

where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the digamma function and the sum is taken over the positive roots of \( \Lambda_l(x, \alpha_c^{(l)}) \)
It is an excellent approximation to neglect the sum over $k$: dashed line vs exact solid line. Other results:

\[ q_{\text{glob}}^{(l)} R_* \big|_{\text{threshold}} \sim \frac{1 + C}{l^3} e^{-3\gamma} \]

\[ q_{\text{glob}}^{(l)} R_* \xrightarrow{\alpha \to \infty} 2(1 + C) e^{\int_0^C dx \left( \frac{1}{C} \frac{1+x}{1-xe^{-x}} - \frac{1}{C-x} \right)} \]
A NUMERICAL STUDY: BEYOND THE GEOMETRIC SPECTRUM
ANALYTICAL VS NUMERICAL FUNCTIONS ($\alpha = 14$)

Solid line: numerical. Dashed line: asymptotic formula common to ($E < 0, R_* = 0$) and ($E = 0, R_* > 0$) analytical solutions. Vertical dotted lines: borders of the matching interval. N.B. $n = 1$ is indeed the ground trimer state.
Relative deviations from geometric spectrum

- at fixed $\alpha$, $\rightarrow 0$ if $n \rightarrow \infty$
- at fixed $n$, $\rightarrow \infty$ if $\alpha \rightarrow \infty$

Experimentally accessible range (due to finite $a$):

- $qR_* > 0.1$ not irrealistic
- On $^6\text{Li}-^40\text{K}$ Feshbach resonance, of width $\Delta B = 1\text{G}$, requires $0.3\text{mG}$ $B$ field stabilization
CONCLUSION

• 2 + 1 fermionic problem, mass ratio $\alpha$, narrow Feshbach resonance

• at each Efimov threshold (of odd angular momentum $l$), the corresponding trimer spectrum is entirely geometric:

\[ E_n^{(l)}(\alpha) \sim E_{\text{glob}}^{(l)} e^{-2\pi n/|s_l|} \quad \forall n \geq 1 \]

where the ground state trimer is $n = 1$

• the exact expression of $E_{\text{glob}}^{(l)}$ shows that it has a finite and non-zero limit at threshold

• opposite limit $\alpha \to +\infty$: spectrum becomes hydrogenoid

\[ E_n^{(l)}(\alpha) \sim \frac{-\hbar^2\alpha}{16\mu R_*^2(n + l)^2} \quad \text{as predicted by the Born-Oppenheimer approximation} \]