Universal behaviour of four-boson systems from a functional renormalisation group

Michael C Birse
The University of Manchester

Results from:

Jaramillo Avila and Birse, arXiv:1304.5454
Schmidt and Moroz, arXiv:0910.4586
Krippa, Walet and Birse, arXiv:0911.4608
Birse, Krippa and Walet, arXiv:1011.5852
Background

Ideas of effective field theory and renormalisation group

- well-developed for few-nucleon and few-atom systems
- rely on separation of scales
- Wilsonian RG used to derive power counting
  → classify terms as perturbations around fixed point (or limit cycle)
Background

Ideas of effective field theory and renormalisation group

- well-developed for few-nucleon and few-atom systems
- rely on separation of scales
- Wilsonian RG used to derive power counting
  → classify terms as perturbations around fixed point (or limit cycle)

Many unsuccessful attempts to extend to dense matter

- but no separation of scales
- other EFT’s for interacting Fermi systems exist
  (Landau Fermi liquid, Ginsburg-Landau theory)
- but parameters have no simple connection to underlying forces
EFTs based on contact interactions

- not well suited for standard many-body methods
  → switch to lattice simulation or look for some more heuristic approach
- based on field theory
- can be matched onto EFT’s for few-body systems
  (input from 2- and 3-body systems in vacuum)

Try functional renormalisation group ("exact" RG)

- based on Wilsonian RG approach to field theories
- successfully applied to various systems in areas from condensed-matter physics to quantum gravity
  [version due to Wetterich (1993)]
Outline

- Functional RG
- Spin-$\frac{1}{2}$ fermions
  - Dimer-dimer scattering
- Bosons
  - Efimov physics
  - 4-body systems
**Functional RG**

Version based on the effective action $\Gamma[\phi_c]$

- start from generating function $W[J]$ defined by

$$e^{iW[J]} = \int D\phi \, e^{i(S[\phi] + J \cdot \phi - \frac{1}{2} \phi \cdot R \cdot \phi)}$$

- $R(q, k)$: regulator function
  suppresses modes with momenta $q \lesssim k$ ("cutoff scale")
- only modes with $q \gtrsim k$ integrated out
- $W[J]$ becomes full generating function as $k \to 0$

Legendre transform $\rightarrow$ effective action

$$\Gamma[\phi_c] = W[J] - J \cdot \phi_c + \frac{1}{2} \phi_c \cdot R \cdot \phi_c \quad \text{where} \quad \phi_c = \frac{\delta W}{\delta J}$$

(generating function for 1-particle-irreducible diagrams)
\[ \partial_k \Gamma = -\frac{i}{2} \text{Tr} \left[ (\partial_k R) \left( \Gamma^{(2)} - R \right)^{-1} \right] \quad \text{where} \quad \Gamma^{(2)} = \frac{\delta^2 \Gamma}{\delta \phi_c \delta \phi_c} \]

\[(\Gamma^{(2)} - R)^{-1}: \text{propagator of boson in background field } \phi_c\]

(one-loop structure but still exact)

Evolution interpolates between “bare” classical action at large scale \( K \) and full 1PI effective action as \( k \to 0 \) (thresholds etc ...)
Γ evolves with scale $k$ according to

$$\partial_k \Gamma = -\frac{i}{2} \text{Tr} \left[ (\partial_k R) \left( \Gamma^{(2)} - R \right)^{-1} \right]$$

where

$$\Gamma^{(2)} = \frac{\delta^2 \Gamma}{\delta \phi_c \delta \phi_c}$$

$$(\Gamma^{(2)} - R)^{-1}$$: propagator of boson in background field $\phi_c$

(one-loop structure but still exact)

Evolution interpolates between “bare” classical action at large scale $K$ and full 1PI effective action as $k \to 0$ (thresholds etc ...)

Functional differential equation

- hard/impossible solve in general
- work with truncated ansatz for $\Gamma$
- local action expanded in powers of derivatives
  (cf low-energy EFTs, but don’t know a priori if we have a consistent power counting)
Derivative expansion may be good at starting scale $K$

- use power counting of EFT to determine relevant terms
  (or use this RG to find that power counting in scaling regime)
- but no guarantee that it remains good for $k \to 0$
  (can’t be for scattering amplitudes at energies above threshold: cuts $\to$ nonanalytic behaviour)
Derivative expansion may be good at starting scale $K$

- use power counting of EFT to determine relevant terms
  (or use this RG to find that power counting in scaling regime)
- but no guarantee that it remains good for $k \to 0$
  (can’t be for scattering amplitudes at energies above threshold:
cuts $\to$ nonanalytic behaviour)
→ need consistency checks:
stability against adding extra terms to ansatz
stability against changes in form of regulator
- use this to optimise choice of regulator [Litim, Pawlowski]
Two species of fermion

Fermion field: $\psi(x)$ (spin-$\frac{1}{2}$ atoms or neutrons)
Boson “dimer” field: $\phi(x)$ (strongly interacting pairs)

Local (nonrelativistic) ansatz for action in vacuum: 2-body sector

$$
\Gamma[\psi, \psi^{\dagger}, \phi, \phi^{\dagger}; k] = \int d^4x \left[ \psi(x)^\dagger \left( i \partial_0 + \frac{\nabla^2}{2M} \right) \psi(x) 
+ Z_\phi(k) \phi(x)^\dagger \left( i \partial_0 + \frac{\nabla^2}{4M} \right) \phi(x) - u_1(k) \phi(x)^\dagger \phi(x) 
- g \left( \frac{i}{2} \phi(x)^\dagger \psi(x)^T \sigma_2 \psi(x) + H c \right) \right]
$$

$g$: AA → D coupling
$u_1(k)$: dimer self-energy ($u_1/g^2$: only physical parameter)
$Z_\phi(k)$: dimer wave-function renormalisation
Evolution equation

\[ \partial_k \Gamma = + \frac{i}{2} \text{Tr} \left[ (\partial_k R_F) \left( (\Gamma^{(2)} - R)^{-1} \right)_{FF} \right] - \frac{i}{2} \text{Tr} \left[ (\partial_k R_B) \left( (\Gamma^{(2)} - R)^{-1} \right)_{BB} \right] \]

\( \Gamma^{(2)} \): matrix of second derivatives of the action

(Gorkov-like form: \( \psi \) and \( \psi^\dagger \) as independent variables → factors of \( \frac{1}{2} \))

“Skeleton” diagram for driving terms in evolution of 2-body parameters

(need to insert \( \partial_k R_F \) on one internal line)

Expand in powers of energy → \( \partial_k u_1, \partial_k Z_\phi \)
3-body sector: AD contact interaction

\[ \Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger; k] = \cdots - \lambda(k) \int d^4x \psi^\dagger(x)\phi^\dagger(x)\phi(x)\psi(x) \]

Evolution of \( \lambda \) driven by terms corresponding to skeletons

- AD contact interaction
- single-A exchange between dimers
  (cf Faddeev and STM equations)
4-body sector: DD→DD, DD→DAA, DAA→DAA terms
[Birse, Krippa and Walet (2010); cf Schmidt and Moroz (2009): bosons]

\[
\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger; k] = \cdots - \int d^4x \left[ \frac{1}{2} u_2(k) (\phi^\dagger \phi)^2 \\
+ \frac{1}{4} v(k) (i \phi^\dagger \phi \psi^T \sigma_2 \psi + H c) \\
+ \frac{1}{4} w(k) \phi^\dagger \phi \psi^\dagger T \psi T \sigma_2 \psi \right]
\]

- dimer “breakup” terms allow 3-body physics to feed in properly
  (cf Faddeev-Yakubovski)
→ coupled evolution equations for \( u_2, v, w \) (27 distinct skeletons)
Regulators

- fermions: sharp cutoff

\[ R_F(q, k) = \frac{k^2 - q^2}{2M} \theta(k - q) \]

- pushes states with \( q > k \) up to energy \( k^2 / 2M \)
- nonrelativistic version of “optimised” cutoff [Litim (2001)]
- fastest convergence at this level of truncation

- bosons

\[ R_B(q, k) = Z_\phi(k) \frac{(c_B k)^2 - q^2}{4M} \theta(c_B k - q) \]

- \( c_B \): relative scale of boson cutoff
- optimised choice \( c_B = 1 \) [cf Pawlowski (2007)]
  (no mismatch between fermion and boson cutoffs)

Also examined smooth cutoffs – more convenient in dense matter
Initial conditions

As $k \to \infty$ boson field purely auxiliary

- $Z_\phi(k) \to 0$
- $u_1(K)$ chosen so that in physical limit $(k \to 0)$

$$u_1(0) = -\frac{M g^2}{4\pi a_0} \quad a_0: \text{AA scattering length}$$

- other couplings $\lambda, u_2, v, w$ also vanish as $k \to \infty$

→ either set $Z_\phi(K) = 0$ etc at large starting scale $K$
or match on to $K^{-n}$ behaviour in scaling regime $K \gg 1/a_0$
Initial conditions

As \( k \to \infty \) boson field purely auxiliary

- \( Z_{\phi}(k) \to 0 \)
- \( u_1(K) \) chosen so that in physical limit \( (k \to 0) \)

\[
u_1(0) = -\frac{M g^2}{4\pi a_0}
\]
a_0: AA scattering length

- other couplings \( \lambda, u_2, v, w \) also vanish as \( k \to \infty \)

\( \to \) either set \( Z_{\phi}(K) = 0 \) etc at large starting scale \( K \)
or match on to \( K^{-n} \) behaviour in scaling regime \( K \gg 1/a_0 \)

Expansion point for \( a_0 > 0 \): dimer binding energy \( \mathcal{E}_D = -1/(Ma_0^2) \)

- external boson lines carry \( P_0 = \mathcal{E}_D \)
- external fermion lines carry \( P_0 = \mathcal{E}_D/2 \)

(below all thresholds)
Results: DD scattering length

- black: “minimal” action – only two-body and DD vertex \( u_2 \)
- red adds three-body coupling \( \lambda \)
- green: full local four-body action, includes \( v, w \)
- purple: similar but using smooth cutoff
Comments

- results seem to converge as more terms are included
- converge to value only weakly dependent on cutoff
  (very little variation over range \(0 \leq c_B \lesssim 2\))
- stationary very close to expected “optimum” \(c_B = 1\)
- incomplete actions \(\rightarrow\) strong dependence on \(c_B\) around \(c_B = 1\)

Final result

\[
\frac{a_B}{a_0} \approx 0.58 \pm 0.02
\]

agrees well with full few-body result

\[
\frac{a_B}{a_0} = 0.6
\]

[Petrov, Salomon and Shlyapnikov (2004)]
Comments

- results seem to converge as more terms are included
- converge to value only weakly dependent on cutoff
  (very little variation over range $0 \leq c_B \lesssim 2$)
- stationary very close to expected “optimum” $c_B = 1$
- incomplete actions $\rightarrow$ strong dependence on $c_B$ around $c_B = 1$

Final result

- $a_B/a_0 \simeq 0.58 \pm 0.02$
- agrees well with full few-body result $a_B/a_0 = 0.6$
  
  [Petrov, Salomon and Shlyapnikov (2004)]
Bosons

More interesting: Efimov effect in 3-body sector

Very similar action and evolution equations

- 3-body coupling $\lambda$ periodic under scaling $k$ by factor $e^{\pi/s_0}$
  where $s_0 = 0.92503$ [Schmidt and Moroz (2009)]
- agrees with Efimov $s_0 = 1.00624$ to $< 10\%$
- no sign of 4-body bound states at this truncation
  [numerical integration requires some care – poles in $\lambda$]

Introduce trimer field $\chi(x)$

- include energy dependence associated with 3-body bound states
- obtain equations with structure like Faddeev-Yakubovsky
  [coupled DD, AT channels]
Effective action

\[
\Gamma_k[\psi, \psi^*, \phi, \phi^*, \chi, \chi^*] = \int d^4 x \left[ \psi^* \left( i \partial_0 + \frac{\nabla^2}{2m} \right) \psi + Z_d \phi^* \left( i \partial_0 + \frac{\nabla^2}{4m} \right) \phi + Z_t \chi^* \left( i \partial_0 + \frac{\nabla^2}{6m} \right) \chi 
- u_d \phi^* \phi - u_t \chi^* \chi - \frac{g}{2} (\phi^* \psi \psi + \psi^* \psi^* \phi) - h (\chi^* \phi \psi + \phi^* \psi^* \chi) 
- \lambda \phi^* \psi^* \phi \psi 
- \frac{u_{dd}}{2} (\phi^* \phi)^2 - \frac{v_d}{4} (\phi^* \phi \phi \psi \psi + \phi^* \psi^* \psi^* \phi) - \frac{w}{4} \phi^* \psi^* \psi^* \phi \psi \psi 
- u_{tt} \chi^* \psi^* \chi \psi - \frac{u_{dt}}{2} (\phi^* \phi \chi \psi + \chi^* \psi^* \phi \phi) 
- \frac{v_t}{2} (\phi^* \psi^* \psi^* \psi \psi + \chi^* \psi^* \phi \psi \psi) \right]
\]
AD interaction $\lambda$ regenerated by evolution even if zero initially [unlike AA scattering]

→ introduce running trimer field [cf Gies and Wetterich (2002)]

$$\partial_k \chi = \zeta_1 \phi \psi + \zeta_2 \psi^\dagger \chi \psi + \zeta_3 \psi^\dagger \phi \phi + \zeta_4 \psi^\dagger \phi \psi \psi$$

where $\zeta_1 = -\partial_k \lambda / 2h$ to cancel running of $\lambda$

• other terms do same for four-atom couplings $v_d$, $w$ and $v_t$

• additional piece in evolution equation

$$\partial_k \Gamma = -\frac{i}{2} \text{Tr} \left[ (\partial_k R) \left( (\Gamma^{(2)} - R)^{-1} \right) \right] + \frac{\delta \Gamma}{\delta \chi} \cdot \partial_k \chi$$

• can keep $\lambda$, $v_d$, $w$ and $v_t = 0$ for all $k$
3-body sector

Coupled equations for $u_t(k)$, $Z_t(k)$ and $h^2(k)$

Scaling limit $k \gg 1/|a_0|$

- couplings oscillate sinusoidally with $\ln k$
- poles in AD scattering amplitude $h^2/u_t$
  (values of $k$ where 3-body bound states appear at zero energy)
  $\rightarrow$ tower of Efimov states with $s_0 = 0.92503$
  momentum scale factor $e^{\pi/s_0} = 29.2$ (exact: 22.7)
- tower cuts off when $k \sim 1/a_0$
4-body sector

3-body couplings $\rightarrow$ cyclic behaviour in $u_{dd}(k)$, $u_{dt}(k)$, and $v_t(k)$

One Efimov cycle of rescaled $\hat{u}_{tt}(k)$ as a function of $t = \ln(k/K)$

solid: real, dashed: imaginary
vertical grey line: AT threshold passes through zero energy
Comments

- imaginary part appears at AT threshold $t = t_3 \simeq -4.85$
- 4-body bound states below AT threshold $t \simeq -3.83, -4.67, \ldots$
  (decay to deeper trimer + free atom $\rightarrow$ finite widths)
- unphysical singularity from zero of $h^2(k)$ at $t \simeq -3.0$
  (end of region within cycle where $h^2(k), Z_t(k)$ have opposite signs $\rightarrow$ trimer “ghost”)


Infinite tower of 4-body bound states below AT threshold

Double exponential pattern $\sim$ super-Efimov effect

[Nishida, Moroz and Son (2013)]

- but may not survive in physical limit $k \rightarrow 0$
- 4-body states may move relative to AT threshold, become virtual
Final cycle of $\hat{u}_{tt}(k)$ for finite $a_0 < 0$

tuned so that last three-body state appears at $k = 0 \ (t = -\infty)$

Three 4-body states, at $t = -4.1, -5.6$ and $-7.1$

(consistent with theorem of Amado and Greenwood)
AA scattering length corresponding to zero-energy 3-body state: \( a_3 \)  
(results just shown)  
Further decrease in \( a \): 4-body states cross zero energy at  
\[
\frac{a_{4}^{(0)}}{a_3} \simeq 0.438, \quad \frac{a_{4}^{(1)}}{a_3} \simeq 0.877, \quad \frac{a_{4}^{(2)}}{a_3} \simeq 0.9967
\]

Two lowest states: ratios within 5% of exact results  
[von Stecher it et al (2009); Deltuva (2010)]

Third state extremely weakly bound  
- if real: challenge to observe numerically and experimentally  
- could be artefact of truncation  
  (Efimov cycles too long: scale factor \( \sim 30 \) instead of 23)
Scaling regime

Scales at which 3- and 4-body states appear

- double exponential form

\[ k_4^{(n)} = k_3 \exp[\alpha e^{-\beta n}] \]

- Ratios of scales given by universal relation

\[ \frac{k_4^{(n+1)}}{k_4^{(n)}} = \left( \frac{k_3}{k_4^{(n)}} \right)^{1 - \exp(-\beta)} \]

Similar to universal scaling function found by Hadizadeh et al. but

- different functional form
- no new 4-body scale parameter
  \((\alpha \text{ fixed, independent of the initial conditions})\)
Summary

Applications of functional RG to 3- and 4-body systems

- local effective action, “optimised” cutoff
- keeping all local terms in 4-body sector

Fermions

- results for dimer-dimer scattering length
  stable against variation of cutoff, agree with direct calculations

Bosons

- dynamical trimer field to match structure of Faddeev-Yakubovsky equations
- imaginary parts of 4-body couplings from each AT threshold
- infinite tower of 4-body bound states below each AT threshold
- double exponential (super-Efimov) pattern
- finite 2-body scattering length: three states in last cycle
Super-Efimov effect

Relies on being close to fixed point with complex scaling
Example for fewer-body coupling $g^2$ at nontrivial fixed point

$$\frac{dv}{dt} = a g^4 + b g^2 v + c v^2$$

with $b^2 - 4ac < 0 \rightarrow$ imaginary scaling dimension

Now consider $g^2$ marginal: $g^2 = g^2_0 / t$ with $t = \ln(k/k_0)$
and define $\hat{v} = t v$

$$t \frac{d\hat{v}}{dt} = a g^4_0 + (1 + b g^2_0) \hat{v} + c \hat{v}^2$$

$\rightarrow$ cyclic behaviour in $\ln t = \ln(\ln(k/k_0))$ if

$$\left( \frac{1}{g^2_0} + b \right)^2 - 4ac < 0$$

4 bosons – close to AAD Efimov cycle [Deltuva (2012)]