Interaction vs correlation effects in many-body systems

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Outline

- Defining correlations: quite an elusive issue
  - correlations in the absence of interaction
  - interaction without correlations

- Theoretical description of correlations
  - particle and hole propagation in interacting many-body systems

- Empirical evidence of nucleon-nucleon correlations
  - nucleon knockout processes
  - Final State Interactions (FSI) in inclusive processes
  - the EMC effect

- Summary & Outlook
Defining correlations

- Consider a system of $N$ interacting particle described by the wave function $\Psi(x_1, \ldots, x_N)$, with $x_i \equiv (r_i, \sigma_i)$
- Probability of finding particles 1, $\ldots$, $n$ at positions $r_1, \ldots, r_n$

$$\rho^{(n)}(r_1, \ldots, r_n) = \frac{N!}{(N-n)!} \sum_{\sigma_1, \ldots, \sigma_N} \int dr_{n+1} \ldots dr_N |\Psi(x_1, \ldots, x_N)|^2$$

- Particles 1 and 2 are correlated if

$$\rho^{(2)}(r_1, r_2) \neq \rho^{(1)}(r_1)\rho^{(1)}(r_2)$$

- The quantity

$$g(r_1, r_2) = \frac{\rho^{(2)}(r_1, r_2)}{\rho^{(1)}(r_1)\rho^{(1)}(r_2)}$$

provides a measure of correlations in coordinate space
The archetype corelated system: the Van der Waals liquid

- Equation of state at particle density $\rho$ and temperature $T$

$$P = \frac{\rho T}{1 - \rho b} - a\rho^2,$$

- $b \propto d^3$ is the “excluded volume”
- $a \sim$ integral of the attractive part of the interaction

- The full Van der Waals potential provides a good description of atomic systems. However, its use in perturbation theory involves non trivial problems.
Enter Pauli’s principle. Consider the ground state of a translationally invariant fermion system at density \( \rho = N/V = k_F^3/3\pi^2 \)

\[
\Psi_0(x_1, \ldots, x_N) = \frac{1}{N!} \det \left[ \phi_{\alpha_i}(x_i) \right], \quad \phi_{\alpha_i}(x_i) = \frac{1}{V^{1/2}} e^{i\mathbf{k} \cdot \mathbf{r}_i} \chi_{\sigma_i}, \quad |\mathbf{k}_i| < k_F
\]

Statistical correlations are described by the function

\[
gr_{FG}(r) = \frac{\rho^{(2)}(r)}{\rho^2} = 1 - \frac{1}{2} \ell^2(k_F r)
\]

\[
\ell(x) = 3 \frac{\sin x - x \cos x}{x^3}
\]
Coordinate vs momentum space

- Bottom line: correlations are best defined in coordinate space.
- To see this, consider the non interacting Fermi gas again. The joint probability of finding two particles with momenta $k_1$ and $k_2$ is

$$ n_{FG}(k_1, k_2) = \theta(k_F - |k_1|)\theta(k_F - |k_2|) \left[ 1 - \frac{1}{N} \frac{\rho}{2} (2\pi)^3 \delta(k_1 - k_2) \right] $$

- In the absence of long range order, a similar result holds true in interacting systems

$$ n(k_1, k_2) = n(k_1)n(k_2) \left[ 1 + O (1/N) \right] $$

- In momentum space, non trivial correlations effects on $n(k_1, k_2)$ vanish in the $N \rightarrow \infty$ limit. However, correlations strongly affect the behaviour of $n(k)$ at $|k| > k_F$. 
Dynamical correlations are induced by two-body interactions described by the potential $v_{ij}$ appearing in the N-particle Hamiltonian

$$H = \sum_{i=1}^{N} -\frac{\nabla^2_i}{2m} + \sum_{j>i=1}^{N} v_{ij},$$

The mean field approximation is based on the replacements

$$\sum_{j>i=1}^{N} v_{ij} \rightarrow \sum_{i=1}^{N} U_i, \quad H \rightarrow \sum_{i=1}^{N} h_i = \sum_{i=1}^{N} \left(-\frac{\nabla^2_i}{2m} + U_i\right),$$

implying

$$H|\Psi_0\rangle = E_0|\Psi_0\rangle \rightarrow h_i|\phi_i\rangle = \epsilon_i|\phi_i\rangle.$$
Within the mean field approximation

\[ E_0 = \sum_{i \in \{F\}} \epsilon_i, \quad \Psi(x_1, \ldots, x_N) = \det[\phi_i(x_i)] \]

\[ \rho^{(2)}(r_1, r_2) = \sum_{i,j \in \{F\}} \phi_i^\dagger(r_1) \phi_j^\dagger(r_2) \left[ \phi_i(r_1) \phi_j(r_2) - \phi_j(r_1) \phi_i(r_2) \right] \]

The mean field approach provides a remarkably accurate description of a variety of properties of interacting many-body systems. However, one should keep in mind that

- dynamical correlations are not taken into account
- including their effects as corrections to the mean field approximation may be highly misleading, as the definition of the mean field itself is model dependent

Theoretical studies aimed at pinning down the role of correlations should be carried out within *ab initio* many body approaches
Model independent determination of correlations

Definition of Green’s function

\[ iG(x - x') = \langle 0 | T [ \hat{\psi}(x) \hat{\psi}^\dagger(x') ] | 0 \rangle \]

After Fourier transformation (\( \eta = 0^+ \))

\[ G(k, E) = \sum_n \left\{ \frac{|\langle n(N+1)(k)|a_k^\dagger|0_N\rangle|^2}{E - (E_n - E_0) + i\eta} + \frac{|\langle n(N-1)(-k)|a_k|0_N\rangle|^2}{E + (E_n - E_0) - i\eta} \right\} \]

\[ = G_p(k, E) + G_h(k, E) = \int dE' \left[ \frac{P_p(k, E')}{E - E' + i\eta} + \frac{P_h(k, E')}{E + E' - i\eta} \right] \]

Spectral functions of hole and particle states

\[ P_h(k, E) = \sum_n |\langle n(N-1)(k)|a_k|0_N\rangle|^2 \delta(E - E_n + E_0) = \frac{1}{\pi} \text{Im} G_h(k, E) \]

\[ P_p(k, E) = \sum_n |\langle n(N+1)(k)|a_k^\dagger|0_N\rangle|^2 \delta(E + E_n - E_0) = \frac{1}{\pi} \text{Im} G_p(k, E) \]
Analytic structure of the Green’s function

★ In interacting systems, the Green’s function (e.g. for hole states) can be written in terms of the particle self energy $\Sigma(k, E)$

$$G_h(k, E) = \frac{1}{E - |k|^2/2m - \Sigma(k, E)}$$

★ Landau’s quasiparticle picture: isolate contributions of 1$h$ (bound) intermediate states, exhibiting poles at energies $\epsilon_k$, given by

$$\epsilon_k = |k|^2/2m + \text{Re} \Sigma(k, \epsilon_k), \text{ as } \text{Im} \Sigma(k, E) \to 0 \text{ (Fermi surface)}$$

★ The resulting expression is

$$G_h(k, E) = \frac{Z_k}{E - \epsilon_k - iZ_k \text{Im} \Sigma(k, \epsilon_k)} + G^B_h(k, E)$$

where $Z_k = |\langle -k|a_k|0\rangle|^2$, and $G^B_h(k, E)$ is a smooth contribution, arising from $2h - 1p, 3h - 2p, \ldots$ (continuum) intermediate states
Correlated Basis Functions (CBF) approach

★ Correlated states obtained from Fermi gas states through the transformation

\[ |n⟩ = \frac{F}{⟨n_{FG}|F^†F|n_{FG}⟩} |n_{FG}⟩ , \quad F = S \prod_{j>i} f_{ij} \]

★ The two-nucleon correlation operator reflects the complexity of the nucleon-nucleon (NN) force [spin-isospin (ST) dependent, non central]

\[ f_{ij} = \sum_{TS} \left[ f_{TS}(r_{ij}) + δ_{S1} f_{Ti}(r_{ij}) S_{ij} \right] P_{TS} \]

\[ P_{TS} : \text{spin – isospin projectors} , \quad S_{ij} = σ_i^α σ_j^β \left( 3r_{ij}^α r_{ij}^β - δ^{αβ} \right) \]

★ Shapes of \( f_{TS}, f_{Ti} \) determined from minimization of ground state energy
Split the hamiltonian according to

\[ H = H_0 + H_I \]

\[ \langle m|H_0|n \rangle = \delta_{mn} \langle m|H|n \rangle \quad , \quad \langle m|H_I|n \rangle = (1 - \delta_{mn}) \langle m|H|n \rangle \]

If correlated states have large overlaps with the eigenstates of the hamiltonian, the matrix elements of \( H_I \) are small and perturbation theory can be used to obtain, e.g., the ground state from

\[ |\tilde{0}\rangle = \sum_m (-)^m \left( \frac{H_I - \Delta E_0}{H_0 - E_0^V} \right)^m |0\rangle \]

\[ \Delta E_0 = E_0^V - E_0 = \langle 0|H|0 \rangle - E_0 \]
Hole spectral function of nuclear matter from CBF

\[ P_h(k, E) = \frac{1}{\pi} \frac{Z_k^2 \text{Im} \Sigma(k, \epsilon_k)}{[E - k^2/2m - \text{Re} \Sigma(k, E)]^2 + [Z_k \text{Im} \Sigma(k, \epsilon_k)]^2} + P_h^B(k, E) \]
Spectral function of infinite nuclear matter

- Results obtained using CBF perturbation theory and the U14+TNI hamiltonian

- The correlation contribution can be identified by its distinctive energy dependence
In analogy with the spectral function, the momentum distribution can be split into quasi particle (pole) and correlation (continuum) contributions

\[ n(k) = \int dE P(k, E) = Z_k \theta(k_F - |k|) + \int dE P_B(k, E) = Z_k \theta(k_F - |k|) + n_B(k) \]
Exploiting the (near) universality of correlations

★ Local density approximation

\[ P(k, E) = P_{MF}(k, E) + P_{corr}(k, E) \]

- \( P_{MF}(k, E) \rightarrow \text{from } (e, e'p) \text{ data} \)

\[ P_{MF}(k, E) = \sum_n Z_n |\phi_n(k)|^2 F_n(E - E_n) \]

- \( P_{corr}(p, E) \rightarrow \text{from uniform nuclear matter calculations at different densities:} \)

\[ P_{corr}(k, E) = \int d^3r \rho_A(r) P_{corr}^{NM}(k, E; \rho = \rho_A(r)) \]

★ Widely and successfully employed to analyze \((e, e')\) data at beam energies \(\sim 1GeV\)

★ Warnings: model dependence, chance of double counting
Theory vs data \((E_e = 1.3 \text{ GeV}, \theta_e = 37.5^\circ)\)

★ Note: calculations involve no adjustable parameters

★ The measured x-section can be described, except in the dip region, between the quasi elastic and \(\Delta\)-production peaks, and the low energy loss tail, where FSI (not included) play a role
Correlation effects on the nuclear response

Consider scattering of a scalar probe, for simplicity

\[
\frac{d\sigma}{d\Omega d\omega} \propto S(q, \omega) = \sum_n \langle 0 | \rho^+_q | n \rangle \langle n | \rho_q | 0 \rangle \delta(E_0 + \omega - E_n)
\]

\[
\rho_q = \sum_k a^+_k q a^+_{k+q} , \quad H|0\rangle = E_0|0\rangle , \quad H|n\rangle = E_n|n\rangle
\]

★ Rewrite the response in the form

\[
S(q, \omega) = \sum_n \left| \sum_k \langle n | a^+_k q a^+_{k+q} | 0 \rangle \right|^2 \delta(\omega + E_0 - E_n)
\]

\[
= \int dt \frac{e^{i(\omega+E_0)t}}{2\pi} \sum_{p,k} \langle 0 | a_{p+q}^+ a^+_{k+q} e^{-iHt} a_{k+q} a_k | 0 \rangle
\]

★ \( S(q, \omega) \) can be expressed in terms of interactions and Green functions describing nucleons in particle and hole states
Effects of interactions on the nuclear response

★ In the absence of correlations, the only possible final states are one particle-one hole states

★ For example, according to the Fermi gas model

\[ M_n = \langle n | \sum_k a_{k+q}^+ a_k | 0 \rangle \rightarrow M_k = 1 \times \theta(k_F - |k|)\theta(|k + q| - k_F) \]

\[ S(q, \omega) = \sum_k |M_k|^2 \delta(\omega + e_0(k) - e_0(k + q)), \quad e_0(k) = \frac{k^2}{2m} \]

★ Inclusion of interactions, through the replacement of Fermi gas states with CBF states, leads to a quenching of the transition matrix elements \( M_k \) and to a modification of the single particle spectrum \( e_0(k) \)
Correlations & interaction effects

★ Isospin symmetric nuclear matter at equilibrium density

★ Correlations

\[ M_{ph} < 1 \]

★ Mean field

\[ m \rightarrow m^* = \left( \frac{1}{k \frac{de}{dk}} \right)^{-1} \]

★ Note that \( m^*(k) \neq m \) does not measure correlation effects
Correlation & interaction effects on the response

\( (A), (B), (C) \rightarrow |q| = 0.3, 1.8, 3.0 \text{ fm}^{-1} \)
Empirical evidence of correlation effects

- Energy dependence of the spectroscopic strengths of shell model states of $^{208}Pb$, measured in high resolution $(e, e'p)$ experiments at NIKHEF-K

- Theory: CBF nuclear matter results corrected for surface effects
The correlation strength in the 2p2h sector has been measured by the JLAB E97-006 Collaboration using a carbon target.

**Strong energy-momentum correlation:**

\[ E \sim E_{thr} + \frac{A-2}{A-1} \frac{k^2}{2m} \]

- Measured correlation strength 0.61 ± 0.06, to be compared with the theoretical predictions 0.64 (CBF) and 0.56 (G-Matrix)
FSI in the impulse approximation regime

- At momentum transfer $|q|^{-1} \gg 2\pi/d$, $d$ being the average interparticle separation distance

$$ S(q, \omega) = \int d^3k dE \, P_h(k, E)P_p(k + q, \omega - E) $$

  - $P_h \to$ many-body theory
  - $P_p \to$ many-body theory + eikonal approximation (OB, arXiv:1301.3357)

- The struck particle travels along a straight trajectory with constant speed $v$. Its propagation is described by the Green’s function ($p = |k + q|$)

$$ G(r_\perp, z) = -\frac{i}{v} \delta(r_\perp)\theta(z) \exp \left[ ipz - \frac{i}{v} \int_0^z d\zeta \, V(\zeta) \right] $$

with

$$ V(\zeta) = \langle 0 | \sum_{j=2}^N \Gamma_p(r_{1j} + \hat{z}\zeta) | 0 \rangle $$
Correlation effects in FSI

★ The interaction is described by the Fourier transform of the scattering amplitude

\[ \Gamma_p(r) = -\frac{2\pi}{m} \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot r} f_p(k) . \]

with

\[ f_p(k) = \frac{p}{4\pi} \sigma_p(\alpha_p + i) e^{-\beta_p k^2} \]

★ FSI are driven by the quantity

\[ V(\zeta) = \int d^3 r \, g(r) \, \Gamma_p(r + \hat{z}\zeta) \]

★ Under the assumptions underlying the eikonal approximation, correlations in coordinate space strongly affect the energy dependence of the spectral function.
Consider the simple case $\alpha_p = \beta_p = 0$, i.e.

$$\text{Im } \Gamma_p(r) = -\frac{1}{2} \rho \nu \sigma_p \delta(r)$$

The corresponding eikonal phase is

$$W(z) = \int_0^z d\zeta V(\zeta) = \frac{1}{2} \rho \sigma_p \int_0^z d\zeta g(\zeta)$$

After Fourier transformation, the $z$-dependence of $W$ leads to a specific energy dependence of the eikonal spectral function.
Preliminary results

★ Isospin symmetric nuclear matter at equilibrium density

★ Main elements of the calculation
  ▶ medium modified nucleon-nucleon cross sections
  ▶ nucleon radial distribution function, $g(r)$
Nuclear binding, correlations and the EMC effect

- The analysis of the dependence of the slope of the EMC ratio on the average nucleon removal energy, defined as

\[
\langle E \rangle = \int d^3k dE P(k, E)
\]

requires a level of accuracy not yet achieved for nuclei with \( A > 3 \).

- Green’s Function Monte Carlo (GFMC) calculations provide the ground state energies, \( E_0 \), and the expectation values of the kinetic energy operator, \( \langle T \rangle \), of nuclei with \( A \leq 12 \), obtained from state-of-the-art nuclear hamiltonian.

- The corresponding average removal energies can be calculated using the GFMC results and the Koltun sum rule, stating that (up to a small correction arising from the three-body potential)

\[
\frac{E_0}{A} = \frac{1}{2} \left[ \frac{A - 2}{A - 1} \langle T \rangle - \langle E \rangle \right]
\]
The slope is analyzed in terms of the variable

\[ \tilde{y} = \nu - |q| \]

that can be interpreted as the longitudinal momentum of the struck particle in the target rest frame. Note that \( \tilde{y} \) is trivially related to Nachtmann’s variable through \( \tilde{y} = -\xi/m \).
The data shows an excellent correlation with $\langle E \rangle$

The analysis includes the ratio obtained from the extrapolated nuclear matter data. The corresponding removal energy is obtained from the values of $E_0$ and $\langle T \rangle$ resulting from the CBF calculation of Akmal & Pandharipande.

The values of $\langle E \rangle$ employed in the analysis are significantly larger than those used in similar studies. For example, in Carbon the removal energy extracted from $(e, e'p)$ data, corresponding to the shell model states, is $\sim 25$ MeV, to be compared to the GFMC result $\sim 52$ MeV.

The large values of $\langle E \rangle$ are to be ascribed to strong nucleon-nucleon correlations, leading to the excitation of nucleons to states of high removal energy and high momentum.
It is long known that correlation effects in nuclei are large. Back in 1952 AD, Blatt & Weiskopf pointed out that:

“The limitation of any independent particle model lies in its inability to encompass the correlation between the positions and spins of the various particles in the system”

While being best defined in coordinate space, correlations manifest themselves in a distinctive energy dependence of the Green’s functions.

Pinning down pure correlation effects in a model independent fashion requires the calculation of the Green’s function within ab initio many-body approaches.

There is ample empirical evidence of important correlation effects from electron-nucleus scattering data. However, the definition of correlation observables remains somewhat elusive.