Tackling the Sign Problem of Ultracold Fermi Gases with Mass-Imbalance

Dietrich Roscher


Advances in quantum Monte Carlo techniques for non-relativistic many-body systems

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Physics and Versatility of Ultracold Fermi Gases

Why we should care about physics of cold gases:

- “Simple” systems compared to e.g. QCD or nuclear physics, but similar behaviour in certain regimes (e.g. HIC) [e.g. A. Adams et al. ’12]
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[C.A.R Sa de Melo, ’08]
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S. Jochim, Uni Heidelberg

[M. Randeria, E. Taylor '08]
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Why “simple” does not mean “easy”:

- Lack of small parameter in strongly interacting regimes invalidates naïve perturbation theory
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Why "simple" does not mean "easy":

- Lack of small parameter in strongly interacting regimes invalidates naïve perturbation theory
- Monte Carlo calculations often severely hampered by sign problems
Idea of the Talk

How to get rid of (some) sign problems:

- Identify problematic quantity $x_s$ and change into $i \cdot x_s$
- Do an *Auxiliary Field Quantum Monte Carlo* calculation without sign problem
- Analytically continue results from $i \cdot x_s$ to physical value $x_s$
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This idea has been proposed for lattice QCD at finite chemical potential some time ago. [M. Alford, A. Kapustin, F. Wilczek ’98; P. de Forcrand, O. Philipsen ’02] Adapting this approach to imbalanced ultracold Fermi gases will be the main subject of this talk.
Describing the Unitary Fermi Gas

Action of a two component 3D Fermi gas with contact interaction:

\[
S[\psi^{\uparrow}, \psi^{\downarrow}] = \int_0^\beta d\tau \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \psi^*_\sigma \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_\sigma + \bar{g} \psi^*_\uparrow \psi^*_\downarrow \psi\downarrow \psi^\uparrow \right\}
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\]

The bare coupling \(\bar{g}\) is given by

\[
\bar{g}^{-1} = \Lambda g^{-1} = \frac{1}{8\pi} \left( a_s^{-1} - c_{\text{reg}}\Lambda \right)
\]

with UV-cutoff \(\Lambda\) and two-body scattering length \(a_s\).

Unitary regime: \(a_s^{-1} \to 0\)
Symmetry and Spontaneous Symmetry Breaking

\[ S[\psi^\uparrow, \psi^\downarrow] = \int_0^\beta d\tau \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^* \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_\sigma + \bar{g} \psi_\uparrow^* \psi_\downarrow^* \psi_\downarrow \psi_\uparrow \right\} \]

\[ U(1)\text{-Symmetry of the action:} \]

\[ \psi^\uparrow,\downarrow \rightarrow e^{i\alpha} \psi^\uparrow,\downarrow; \quad \psi^*_{\uparrow,\downarrow} \rightarrow \psi^*_{\uparrow,\downarrow} e^{-i\alpha} \]

Spontaneously broken if \( \langle \psi_\downarrow \psi_\uparrow \rangle \neq 0 \)
Symmetry and Spontaneous Symmetry Breaking

\[ S[\psi^\uparrow, \psi^\downarrow] = \int_0^\beta d\tau \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \psi^{\ast\sigma} \left( \partial_{\tau} - \frac{\nabla^2}{2m} - \mu \right) \psi_{\sigma} + \bar{g}\psi^{\ast\uparrow}\psi^{\ast\downarrow} \psi^\downarrow \psi^\uparrow \right\} \]

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Spontaneously broken if \( \langle \psi^\downarrow \psi^\uparrow \rangle \neq 0 \)

\( \implies \) Condensation of bound states

[C.A.R Sa de Melo, '08]
Partition Function and Observables

Partition function and observables from the path integral:

\[ Z = \int D\psi^\uparrow D\psi^\downarrow e^{-S[\psi^\uparrow,\psi^\downarrow]} \quad \Rightarrow \quad \langle \hat{O} \rangle = \frac{1}{Z} \int D\psi^\uparrow D\psi^\downarrow O e^{-S[\psi^\uparrow,\psi^\downarrow]} \]
Partition Function and Observables

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*Hubbard-Stratonovich* transformation and integration of fermion fields:

$$ 1 = \mathcal{N} \int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{-\int d\tau \int d^3 x \ m_\varphi^2 \varphi \varphi^*}, \quad \varphi \rightarrow \varphi - \frac{g_\varphi}{m_\varphi^2} \psi_\uparrow \psi_\downarrow $$
Partition Function and Observables

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*Hubbard-Stratonovich* transformation and integration of fermion fields:

\[ 1 = \mathcal{N} \int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{-\int d\tau \int d^3x \frac{m^2}{2} \varphi \varphi^*}, \quad \varphi \rightarrow \varphi - \frac{g_\varphi}{m^2} \psi_\uparrow \psi_\downarrow \]

\[ Z = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \det \left[ \hat{G}^{-1} + \hat{\Phi} \right] e^{-\int d\tau \int d^3x \frac{m^2}{2} \varphi \varphi^*} \]

\[ \hat{G}^{-1} = \begin{pmatrix} -i\omega_n - \frac{\nabla^2}{2m} - \mu & 0 \\ 0 & i\omega_n - \frac{\nabla^2}{2m} - \mu \end{pmatrix}, \quad \hat{\Phi} = \begin{pmatrix} 0 & g_\varphi \varphi \\ -g_\varphi \varphi^* & 0 \end{pmatrix} \]

⇒ Starting point for actual computations ⇐
Positivity of the Fermion Determinant

\[ Z = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \det \left[ \hat{G}^{-1} + \hat{\Phi} \right] e^{-\int d\tau \int d^3x \ m^2 \varphi \varphi^*} \]

If the fermion determinant is positive for all admissible \( \varphi(x) \), a positive definite probability measure can be defined for Monte Carlo calculations - there is no sign problem.

\[
\det \hat{G}^{-1} \sim \det \begin{pmatrix} \hat{A} & 0 \\ 0 & \hat{A}^* \end{pmatrix} = \det \left[ \hat{A} \hat{A}^* \right] \geq 0
\]

Since the interaction does not break the symmetry between \( \uparrow \) and \( \downarrow \) fermions, the symmetry of the eigenvalue distribution is conserved for finite \( \varphi \).

Thus, the fermion determinant is positive for all \( \varphi(x) \).
Introducing Imbalance

\[
S[\psi^\uparrow, \psi^\downarrow] = \int_0^\beta d\tau \int d^3x \left\{ \sum_{\sigma = \uparrow, \downarrow} \psi^*_\sigma \left( \partial_\tau - \frac{\nabla^2}{2m_\sigma} - \mu_\sigma \right) \psi_\sigma + \bar{g} \psi^*_\uparrow \psi^*_\downarrow \psi^\downarrow \psi^\uparrow \right\}
\]
Introducing Imbalance

\[ S[\psi_{\uparrow}, \psi_{\downarrow}] = \int_{0}^{\beta} d\tau \int d^{3}x \left\{ \sum_{\sigma=\uparrow, \downarrow} \psi_{\sigma}^{\ast} \left( \partial_{\tau} - \frac{\nabla^{2}}{2m_{\sigma}} - \mu_{\sigma} \right) \psi_{\sigma} + \bar{g}\psi_{\uparrow}^{\ast}\psi_{\downarrow}^{\ast}\psi_{\downarrow}\psi_{\uparrow} \right\} \]

Experimental relevance:

- Population imbalance (\(\mu_{\sigma}\)): Tuning of the species relative population via a magnetic field \(h\) [e.g. Zwierlein et al. '06, Partidge et al. '06]

- Mass imbalance (\(m_{\sigma}\)): Mixtures of different elements, e.g. \(^{6}Li\) and \(^{40}K\) [e.g. A. Gezerlis, S. Gandolfi, K.E. Schmidt, J. Carlson '09; K.B. Gubbels, J.E. Baarsma, H.T.C. Stoof '09]

BUT
Introducing Imbalance

\[ S[\psi_\uparrow, \psi_\downarrow] = \int_0^\beta d\tau \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \psi^*_\sigma \left( \partial_\tau - \frac{\nabla^2}{2m_\sigma} - \mu_\sigma \right) \psi_\sigma + \bar{g} \psi^*_\uparrow \psi^*_\downarrow \psi_\downarrow \psi_\uparrow \right\} \]

Population- and/or mass-imbalance destroy the symmetry between \( \uparrow \) and \( \downarrow \) particles and thus also the positivity of the fermion determinant:

\[ \hat{G}^{-1} = \begin{pmatrix} -i \omega_n - \frac{\nabla^2}{m_+} - \frac{\nabla^2}{m_-} - \bar{\mu} - h & 0 \\ 0 & i \omega_n - \frac{\nabla^2}{m_+} + \frac{\nabla^2}{m_-} - \bar{\mu} + h \end{pmatrix} \]

with

\[ \bar{\mu} = \frac{\mu_\uparrow + \mu_\downarrow}{2}, \quad h = \frac{\mu_\uparrow - \mu_\downarrow}{2} \]

\[ m_+ = \frac{4m_\uparrow m_\downarrow}{m_\downarrow + m_\uparrow}, \quad m_- = \frac{4m_\uparrow m_\downarrow}{m_\downarrow - m_\uparrow} \]

Sign Problem!
The Sign Problem and Imaginary Imbalance

The sign problem could be resolved, if the $\uparrow - \downarrow$ symmetry was reinstated. Define complex-valued particle masses $m_\sigma^C$ and chemical potentials $\mu_\sigma^C$ such that:

$$h = ih_I, \quad h_I \in \mathbb{R}, \quad \bar{m} \equiv \frac{m_+^C}{m_-^C} = i\bar{m}_I, \quad \bar{m}_I \in \mathbb{R}$$

The blocks of $\hat{G}_C^{-1}$ are again complex conjugates of each other:

$$\hat{G}_C^{-1} = \begin{pmatrix}
-i\omega_n - \frac{\nabla^2}{m_+} - i\bar{m}_I \nabla^2 - \bar{\mu} - i h_I & 0 \\
0 & i\omega_n - \frac{\nabla^2}{m_+} + i\bar{m}_I \nabla^2 - \bar{\mu} + i h_I
\end{pmatrix}$$
The Sign Problem and Imaginary Imbalance

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Population- and mass-imbalance behave quite differently. For population imbalance, see


For a large part of the talk: purely mass imbalanced case ($h = 0$)
The Sign Problem and Imaginary Imbalance

The sign could be resolved, if the $\uparrow - \downarrow$ symmetry was reinstated. Define complex-valued particle masses $m^C_\sigma$ and chemical potentials $\mu^C_\sigma$ such that:

$$h = i h_l, \quad h_l \in \mathbb{R}, \quad \bar{m} \equiv \frac{m^C_+}{m^C_-} = i \bar{m}_l, \quad \bar{m}_l \in \mathbb{R}$$

The blocks of $\hat{G}^{-1}_C$ are then complex conjugates of each other:

$$\hat{G}^{-1}_C = \begin{pmatrix} -i \omega_n - \frac{\nabla^2}{m_+} - i \bar{m}_l \nabla^2 - \bar{\mu} - i h_l & 0 \\ 0 & i \omega_n - \frac{\nabla^2}{m_+} + i \bar{m}_l \nabla^2 - \bar{\mu} + i h_l \end{pmatrix}$$

Sign Problem Circumvented!!
The Sign Problem and Imaginary Imbalance

The sign could be resolved, if the $\uparrow - \downarrow$ symmetry was reinstated. Define complex-valued particle masses $m_\sigma^C$ and chemical potentials $\mu_\sigma^C$ such that:

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The blocks of $\hat{G}_{\sigma}^{-1}$ are then complex conjugates of each other:

$$\hat{G}_{\sigma}^{-1} = \begin{pmatrix}
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\end{pmatrix}$$

Sign Problem Circumvented!! But...

How to get back to real (physical) imbalance?
Physical Results from Imaginary Calculations

Suppose, some observable has been computed for imaginary mass-imbalance. Then fit the data points with, e.g., some polynomial
\[ \langle \hat{O}^C \rangle \sim \sum_{n=0}^{N_{\text{max}}} C_{\hat{O}}(n) \tilde{m}_I^n \]
and analytically continue \( \tilde{m}_I \rightarrow -i \tilde{m} \).

Schematically:
Physical Results from Imaginary Calculations

Suppose, some observable has been computed for imaginary mass-imbalance. Then fit the data points with, e.g., some polynomial
\[ \langle \hat{O}^C \rangle \sim \sum_{n=0}^{N_{\text{max}}} C^{(n)}_O \bar{m}_I^{2n} \] and analytically continue \( \bar{m}_I \to -i\bar{m} \).

Schematically:

Are there any limits to the method, once Monte Carlo data is available?
Analytic Limits for the Method

Meaningful results can only be expected, if the series representation for $\langle \hat{O}^C \rangle$ converges in the complex $\bar{m}$-plane:

$$\langle \hat{O}^C \rangle(-i\bar{m}_I) = \langle \hat{O} \rangle(\bar{m}) \iff \bar{m}_I \leq r_{\bar{m}}$$

The radius of convergence is determined by the closest singularity.
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Knowledge of $r_{\bar{m}}$ (the singularity structure) is crucial to ascertain reliability of the results, see also examples below.

$\Rightarrow$ Analytical pre-treatment is required
How can $r_{\bar{m}}$ be obtained?

Plan:
- Perform fully analytical calculation $\varphi = \text{const mean-field case}$
- Investigate convergence properties
- Get an idea of the analytic structure of the theory and see what can be used for actual MC data
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Mean-field theory:

Reduce $Z = \int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{-\Gamma[\varphi,\varphi^*]}$ to $Z_{MF} = e^{-\Gamma[\varphi_0,\varphi_0^*]}$ and $\Gamma[\varphi_0,\varphi_0^*] \doteq \text{min}$
How can $r\bar{m}$ be obtained?

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Mean-field theory:

Reduce $Z = \int D\varphi D\varphi^* e^{-\Gamma[\varphi,\varphi^*]}$ to $Z_{\text{MF}} = e^{-\Gamma[\varphi_0,\varphi_0^*]}$ and $\Gamma[\varphi_0, \varphi_0^*] = \text{min}

The grand canonical potential is then just given by

$$\Omega_{\text{MF}} = -\frac{1}{\beta} \ln Z_{\text{MF}} = \frac{1}{\beta} \Gamma[\varphi_0, \varphi_0^*]$$
\( \Omega_{\text{MF}} \) and the Gap Equation

For \( \varphi = \text{const} \), \( \det \left[ \hat{G}^{-1} + \hat{\Phi} \right] \) can be computed analytically by performing a Bogoliubov transformation and the Matsubara sum.

Result (\( m_+ = 1 \), regularizing terms dropped):

\[
\Omega_{\text{MF}} \left[ |\varphi|^2 \right] = \int \frac{d^3 q}{(2\pi)^3} \left\{ \frac{g_\varphi^2 |\varphi|^2}{2q^2} - \frac{1}{\beta} \ln \left[ \cosh \left( \beta \tilde{m} q^2 \right) + \cosh \left( \beta \sqrt{(q^2 - \tilde{\mu}^2)^2 + g_\varphi^2 |\varphi|^2} \right) \right] \right\}
\]
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$$

Gap equation with $\frac{g_\varphi^2 |\varphi_0|^2}{\bar{\mu}^2} \equiv \bar{\Delta}$:

$$
0 \overset{!}{=} \int dq \left\{ \frac{1}{2} + \frac{q^2}{2\sqrt{(q^2 - 1)^2 + \bar{\Delta}}} \left( \frac{1}{\beta \bar{\mu} \left( q^2 \bar{m} + \sqrt{(q^2 - 1)^2 + \bar{\Delta}} \right)} - \frac{1}{1 + e^{\beta \bar{\mu} \left( q^2 \bar{m} - \sqrt{(q^2 - 1)^2 + \bar{\Delta}} \right)}} \right) \right\}
$$
Mean-Field Phase Diagram for the Mass-Imbalanced Unitary Fermi Gas

Position of the critical point: \( \left( \frac{T_{CP}}{\mu}, \bar{m}_{CP} \right) \approx (0.47, 0.37) \)
Comparing Phase Diagrams for $\bar{m}$ and $\bar{m}_I$

Since the functional form of $\Omega_{MF}$ is known analytically, both phase diagrams can be determined directly:

- No critical point/1st order transition for $\bar{m}_I$
- Analytic continuation of phase boundary will not reproduce result for all $\bar{m}$ ⇒ radius of convergence $r_{\bar{m}}$?
Quantitative bounds on $r_{\bar{m}}$

Functional form of $\Omega_{\text{MF}}(\bar{m})$ not explicitly known due to integration:

$$\Omega_{\text{MF}}(\bar{m}) = \int \frac{d^3 q}{(2\pi)^3} \left\{ \frac{g^2 |\varphi|^2}{2q^2} - \frac{1}{\beta} \ln \left[ \cosh \left( \beta \bar{m} q^2 \right) + \cosh \left( \beta \sqrt{(q^2 - \bar{\mu}^2)^2 + g^2 |\varphi|^2} \right) \right] \right\}$$

Complex analysis: $r_{\bar{m}}$ bounded from below by singularity structure of the integrand, i.e. the log term.
Quantitative bounds on $r\bar{m}$

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Complex analysis: $r\bar{m}$ bounded from below by singularity structure of the integrand, i.e. the log term. With $\Phi = g_\varphi \varphi$

$$\Rightarrow \quad r\bar{m} \geq r_{\text{min}} = \sqrt{\frac{\beta^2 |\Phi|^2 + \pi^2}{\beta^2 |\Phi|^2 + \pi^2 + \beta^2 \bar{\mu}^2}}$$
Properties and Usefulness of $r_{\text{min}}$

$$r_{\text{min}} = \sqrt{\frac{\beta^2 |\Phi|^2 + \pi^2}{\beta^2 |\Phi|^2 + \pi^2 + \beta^2 \bar{\mu}^2}}$$

Limiting cases:

- $r_{\text{min}}|_{T \to \infty} = 1$, i.e. access to all possible $\bar{m}$ for high temperatures
- $r_{\text{min}}|_{T \to 0} = \sqrt{\frac{|\Phi|^2}{|\Phi|^2 + \bar{\mu}^2}}$, i.e. (partial) access to symmetry broken phases at $T = 0$
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Applicability:
- If $\Omega^C$ itself is computed for certain $(\bar{m}_I, T)$, $T$ fixed, it can be continued to $\Omega$ inside the interval $[0, r_{\text{min}}(T)]$
  Physical observables can then be obtained from $\Omega$.
- Additional expansions of $\Omega^C$ around some $\bar{m} > 0$ may vastly extend regime of applicability [F. Karbstein, M. Thies '07]
Properties and Usefulness of $r_{\text{min}}$

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Limiting cases:

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  Physical observables can then be obtained from \( \Omega \).
- Additional expansions of \( \Omega^C \) around some \( \bar{m} > 0 \) may vastly extend regime of applicability \( [F. Karbstein, M. Thies '07] \)
- If observables \( \langle \hat{O}^C \rangle \) are computed directly, as it is most often the case with Monte Carlo, things get a little more complicated...
Application I: Phase Boundary

Information provided by $\Omega$:

- Boundary $T_c(\bar{m})$ of phase with broken symmetry is defined for every $\bar{m}$ by the lowest $T$ with $\bar{\Delta} = 0$

- Locally smooth manifold up to critical point, global properties not accessible due to implicit form
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- Locally smooth manifold up to critical point, global properties not accessible due to implicit form
- Analytic continuation of boundary is meaningful if $\Omega[T_c(\bar{m}), \bar{m}, \bar{\Delta} = 0]$ is convergent $\Rightarrow r_{\text{min}}(|\Phi|^2 = 0) = \sqrt{\frac{T^2 \pi^2}{T^2 \pi^2 + \bar{\mu}^2}}$

![Graph showing phase boundary](image-url)
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![Graph showing the phase boundary](image-url)
Phase Boundary from $\bar{m}_I$

Continuation of the $\bar{m}_I$ data yields:

- Fair reproduction of phase boundary up to the critical point
- Non-analytic behaviour of $T_c(\bar{m})$ itself limits applicability
Phase Boundary from $\bar{m}_I$

Continuation of the $\bar{m}_I$ data yields:

- Fair reproduction of phase boundary up to the critical point
- Non-analytic behaviour of $T_c(\bar{m})$ itself limits applicability
- Way out: computation & continuation of $\Omega$ itself, critical point should then be within reach
Application II: Bertsch Parameter $\xi_{T=0}$

Definition:

$$\xi = \frac{\bar{\mu}}{\epsilon_F}, \quad \xi_{T=0}^{\text{Free}} = 1 \quad \forall \bar{m} \in [0, 1)$$
Application II: Bertsch Parameter $\xi_{T=0}$

Definition:

$$\xi = \frac{\bar{\mu}}{\epsilon_F}, \quad \xi_{T=0}^\text{Free} = 1 \quad \forall \bar{m} \in [0, 1)$$

- Smooth observable implicitly depends on $\bar{\Delta}$
- By complex analysis: $r_\xi \leq \min[r_{\text{min}}, r_\Delta]$  
- First order phase transition is expected to limit continuation of $\xi^C$
Good reproduction of $\xi$ up to the $\bar{\Delta}$ discontinuity
Smooth observables indeed limited by $r_{\min}$ or $r_{\Delta}$
Mass-Imbalanced Fermi Gases
Observables & Monte Carlo

Detour: Population-Imbalance

\[ G_{\uparrow}^{-1} = -i\omega_n - \frac{\nabla^2}{m_+} - i\bar{m}_I \nabla^2 - \bar{\mu} - ih_I, \quad \omega_n = (2n + 1)\pi T \]

- Due to interference with Matsubara frequencies: \( h_I < \pi T \)
- Continuation of \( \xi_{T=0} \) not possible for population imbalanced case
- Structurally similar to finite \( \mu \) problem in lattice QCD

Remarks on $r_{\text{min}}$ for Monte Carlo

Schematic structure of the grand canonical potential from Monte Carlo:

$$\Omega_{\text{MC}} \sim \ln \left\{ \sum_{\{\varphi(x)\}} \det \left[ \hat{G}^{-1} + \hat{\Phi} \right] e^{-\int m^2 \varphi(x)^2} \right\}$$

Rigorous statements:

- For every fixed configuration $\varphi(x)$, the corresponding analytical calculation would correspond to the above mean-field procedure.
- Overall radius of convergence: $r_{\text{MC}} \geq \min_{\{\varphi(x)\}}[r_{\text{min}}]$
- Problem: no easy way to calculate $r_{\text{min}}$ for general $\varphi(x)$
Remarks on $r_{\text{min}}$ for Monte Carlo

Schematic structure of the grand canonical potential from Monte Carlo:

$$\Omega_{MC} \sim \ln \left\{ \sum_{\{\varphi(x)\}} \det \left[ \hat{\mathcal{G}}^{-1} + \hat{\Phi} \right] e^{-\int m^2_\varphi |\varphi(x)|^2} \right\}$$

Rigorous statements:

- For every fixed configuration $\varphi(x)$, the corresponding analytical calculation would correspond to the above mean-field procedure.
- Overall radius of convergence: $r_{MC} \geq \min_{\{\varphi(x)\}}[r_{\text{min}}]$.
- Problem: no easy way to calculate $r_{\text{min}}$ for general $\varphi(x)$.

First practical estimate: $r_{MC} \geq r_{\text{min}}(\varphi = 0)$
Summary

- Imbalanced strongly interacting Fermi gases are tough systems to compute with Monte Carlo methods due to a sign problem.
- Imaginary imbalance parameters remove the sign problem.
  - Physical observables may be extracted by analytic continuation.
  - Large parts of the phase diagram of the 3D unitary Fermi gas are in reach.
- Extraction of physical results is limited by convergence issues.
  - Radius of convergence of the grand potential $\Omega$.
  - Analyticity of the observables and dependencies.
  - Type of imbalance.
- Analytic structure at mean-field level provides hints for treatment of genuine Monte Carlo data.
Outlook

- Extend investigation of convergence properties of observables
- Make connection to Monte Carlo more rigorous, ideally provide rigorous quantitative bounds
- Apply method in an actual Monte Carlo study

⇒ Work in progress by Joaquín Drut ⇐