Current noise in topological Josephson junctions

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Motivation

• active search for Majorana fermions in superconducting hybrid structures

proposed signatures:

• zero-energy bound state (visible in tunneling DoS)
• fractional AC Josephson effect?

→ study properties of a biased topological Josephson junction
BEST KNOWN EXAMPLE: QH system

NOVEL ASPECTS:
• time-reversal symmetry ✓
• strong spin-orbit interaction → band inversion

Example:
HgTe/CdTe quantum wells
Koenig et al., Science 421, 766 (2007)

[also InAs/GaSb
Knez-Dju & Sullivan, PRL 107, 136603 (2011)]

X-L Qi & S-C Zhang,
RMP 83, 1057 (2011)
The coefficients and open boundary condition states at the edge, and described by the effective helical edge theory (Wu et al., 2008; Zyuzin and Fiete, 2010). In this context, the concept of "helical" edge states propagates in one direction only, as shown in Fig. 1(a). For HgTe QWs, we have the edge state dispersion relation shown in Fig. 4(b), or the Hamiltonian for the topological edge states. Another important quantity characterizing the edge states is their density of states, which is defined as

\[ \rho(E) = \text{max} \{\Re \lambda(E)\} \]

The sign of the edge states can be determined by the effective helical edge Hamiltonian defined by

\[ H = -i v \partial_x \sigma_z \]

or

\[ \Psi = \begin{pmatrix} \psi_R^\uparrow \\ \psi_L^\downarrow \end{pmatrix} \]

The dashed line stands for a typical value of the chemical potential gap, and the Dirac velocity of the edge states is given by

\[ v = \frac{1}{2} \frac{\lambda}{\mu} \]

Thus, perfect transmission. However, this effect is not robust against the influence of finite width, which can also include the contribution of edge states into the bulk determines the amplitude for inelastic edge tunneling (Hou et al., 2009; Teo and Kane, 2009; Zhour, 2008; Tse et al., 2006). Reference to facts that states with only spin down electrons can propagate backwards, the edge, it can in principle cause backscattering of the helical edge states. If a nonmagnetic impurity is present near the edge, it can scatter electrons either moving clockwise or counterclockwise turn around the impurity [Fig. 5(b)]. Similarly, the spin up on the QSH edge can make either a clockwise or a counterclockwise turn around the impurity. Thus, TR symmetry prevents the helical edge states from undergoing backscattering.

The discussion is continued in the next slide.
2D Topological Insulator (QSH): Helical Edge States

- effective low-energy Hamiltonian: \( \mathcal{H} = -iv\partial_x \sigma_z \)  
  (Dirac fermions)

\[
\Psi = \begin{pmatrix}
\psi_R^+ \\
\psi_L^-
\end{pmatrix}
\]

Koenig et al. 2007
1D Topological Superconductor: Majorana Fermions

- topological superconductor: 1D spinless p-wave
- edge state = Majorana fermion

(a) 

(b) 

Kitaev, Phys. Usp. (2001)

Possible realizations:
- 2D topological insulator (QSH) + conventional superconductor
- nanowire with strong spin-orbit coupling & Zeeman field + conventional superconductor proximity effect → topological superconductivity

Oreg, Refael & von Oppen, PRL (2010)
Lutchyn, Sau & Das Sarma, PRL (2010)

tunnel spectroscopy:
zero-energy bound state at the end of the nanowire in the topological phase

Mourik et al. (2012)
Fractional AC Josephson Effect – Setup

2D Topological Insulator

Superconductor

Superconductor
Fractional AC Josephson Effect – Setup
Andreev / Majorana Bound States

Hamiltonian:
\[ \mathcal{H} = v_p \sigma_z \tau_z + \Delta(x) e^{i\varphi(x)} \tau_z \tau_x + M(x) \sigma_x \]

\[ \delta \varphi = \varphi_2 - \varphi_1 \]
Andreev / Majorana Bound States

- $M = 0$: bound state
  \[ \epsilon(\delta \varphi) = \pm \Delta \cos \frac{\delta \varphi}{2} \]

- backscattering? potential barrier \( \times \)
  magnetic barrier \( H_M = M(x) \sigma_x \)
  \( \rightarrow \) spin-flip scattering \( \checkmark \)

- bound state
  \[ \epsilon(\delta \varphi) = \pm \sqrt{D} \Delta \cos \frac{\delta \varphi}{2} \]
  where $D$ transmission probability

For comparison:
S-N-S with potential barrier
Josephson current carried by ABS:  

\[ I_{\pm}(\delta \varphi) = \pm \frac{e}{2\hbar} \sqrt{D} \Delta \sin \frac{\delta \varphi}{2} \]

**equilibrium Josephson current:**  

\[ I(\delta \varphi) = \frac{e}{\hbar} \frac{\partial \varepsilon_<(\delta \varphi)}{\partial \delta \varphi} \]

where \( \varepsilon_< \) is the energy of the ABS below the Fermi level

\[ I(\delta \varphi) = \frac{e}{2\hbar} \sqrt{D} \Delta \sin \frac{\delta \varphi}{2} \text{sign} \left( \cos \frac{\delta \varphi}{2} \right) \]

\( \rightarrow \) 2\( \pi \)-periodic
Josephson current carried by ABS:  

\[ I_{\pm}(\delta \varphi) = \pm \frac{e}{2\hbar} \sqrt{D\Delta} \sin \frac{\delta \varphi}{2} \]

non-equilibrium Josephson current: \( \delta \varphi \to 2eV_{dc}t \)

\[ I(V_{dc}) = \frac{e}{2\hbar} \sqrt{D\Delta} \sin(eV_{dc}t) \]

in the absence of inelastic processes

- fractional Josephson effect at frequency \( eV_{dc} = \omega_J/2 \)?
- only even Shapiro steps?

\[
\sin \left( eV_{dc}t + \frac{eV_{ac}}{\Omega} \sin \Omega t + \frac{\varphi_0}{2} \right) \neq 0 \quad \text{for} \quad eV_{dc} = k\Omega
\]

(Kitaev, Ann. Phys. (2003))

(Fu & Kane, PRB (2009))

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Outline

• Andreev Bound States: S-N-S vs S-TI-S Josephson Junctions ✓

Fractional AC Josephson effect?

• Effective Model: Bound State Dynamics
  – Characteristic Time Scales
  – Average current & Noise
• Conclusions I

• Scattering Theory
  – Noise: comparison with the effective model
  – Additional signatures → MAR features
• Conclusions II

NOTE: $\delta \phi = \phi = \chi$

$V = V_{dc}$
Main Result

• 2 characteristic time scales:
  - lifetime of the Majorana bound state $\tau_s$
    due to dynamic coupling to the continuum
  - phase adjustment time $\tau_R$
    due to the resistive environment provided by the circuit

observable signatures:

• even/odd effect in Shapiro steps for $\tau_s \gg \tau_R$
• more robust:
  peak in finite-frequency noise $S(\omega)$
  at $\omega = eV_{dc}$ (half of the usual Josephson frequency)
Effective Model

1. switching probability: ‘Landau-Zener’

2. bound state dynamics: Markov process
Switching Probability: ‘Landau-Zener’

at finite bias:

- \( \varphi(t) = 2e \int t' V(t') \, dt' \)
- dynamical coupling between the bound state and the continuum
  \[ \rightarrow \text{occupation of the bound state may switch} \]
Switching Probability: ‘Landau-Zener’

Hamiltonian:
\[ \mathcal{H} = v p \sigma_z \tau_z + \Delta(x) e^{i \varphi(x,t)} \tau_z \tau_x + M(x) \sigma_x \]

consider weak backscattering: \( R = 1 - D \ll 1 \)
where \( R \sim (ML/v)^2 \)

\[ \frac{\epsilon(\chi)}{\Delta} \]

\[ \delta \approx \frac{R}{2} \Delta \ll \Delta \]

\[ \chi(t) \]
Switching Probability at $R \ll 1$

$$\mathcal{H} = v p \sigma_z \tau_z + \Delta(x) e^{i \varphi(x,t) \tau_z} \tau_x + M(x) \sigma_x$$

consider only states with energy near $\Delta$:

$$\varphi \ll 1$$
$$vp \ll \Delta$$

apply unitary transformation & diagonalize particle-hole space

$$\rightarrow \quad H = \Delta + \frac{v^2 p^2}{2\Delta} + v \left[ \frac{1}{2} \varphi \sigma_z + \sqrt{R} \sigma_x \right] \delta(x)$$

bound state similar to Shiba state created by a magnetic impurity in a superconductor

$$\epsilon_A = \Delta \left( 1 - \frac{\varphi^2}{8} - \frac{R}{2} \right)$$

Shiba 1968, Rusinov 1969
Switching Probability at $R \ll 1$

\[ H = \Delta + \frac{v^2 p^2}{2\Delta} + v \left[ \frac{1}{2} \varphi \sigma_z + \sqrt{R} \sigma_x \right] \delta(x) \]

+ time-dependent phase: $\varphi = 2eV_{dc} t$

use adiabatic basis

\[ |\Psi(t)\rangle = c_A(t)|\Psi_A(t)\rangle + \sum_{p,\sigma} c_{p,\sigma}(t)|\Psi_{p,\sigma}(t)\rangle \]

\[ \epsilon(\chi)/\Delta \]

$c_A(-\infty) = 1$

$c_{p,\sigma}(-\infty) = 0$

two-band version of transition from discrete state to continuum

Demkov-Osherov (1967)

switching probability

\[ s = \sum_{p,\sigma} |c_{p,\sigma}(+\infty)|^2 \]
Switching Probability at $R \ll 1$

$$H = \Delta + \frac{v^2 p^2}{2\Delta} + v \left[ \frac{1}{2} \varphi \sigma_z + \sqrt{R} \sigma_x \right] \delta(x)$$

adiabaticity parameter: $\lambda = \frac{R^{3/2} \Delta}{eV_{dc}}$

switching probability

$$s \approx \begin{cases} 
  c_{>} \lambda^{-5/4} e^{-2\lambda/3} & \lambda \gg 1 \\
  1 - c_{<} \lambda^{2/3} & \lambda \ll 1 
\end{cases}$$

time scale for the transition:

$$\tau_t \sim \begin{cases} 
  \sqrt{R/eV_{dc}} & \ll \pi/eV_{dc} \\
  1/[\Delta(eV_{dc})^2]^{1/3} & \ll \pi/eV_{dc}
\end{cases}$$
Switching Probability at $R \ll 1$

$$s \approx 1 - 1.05 \lambda^{2/3}$$

$$s \approx 0.93 \lambda^{-5/4} e^{-2 \lambda/3}$$

$\lambda = R^{3/2} \Delta / (eV_{dc})$
Bound State Dynamics: Markov Process

\[ \chi(t)/2\pi \]

\[ n(t) \]

\[ \epsilon(t) \]

\[ I(t) \]
$P$ probability of fermionic state to be occupied
$Q = 1 - P$ probability of fermionic state to be empty
$s$ switching probability per half-period

$$\begin{pmatrix} P_{2n} \\ Q_{2n} \end{pmatrix} = \begin{pmatrix} 1 - s & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} P_{2n-1} \\ Q_{2n-1} \end{pmatrix}$$

$$\begin{pmatrix} P_{2n+1} \\ Q_{2n+1} \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 - s \end{pmatrix} \begin{pmatrix} P_{2n} \\ Q_{2n} \end{pmatrix}$$

current: $I(t) = I_0 \sin \frac{\varphi(t)}{2} x_{\text{Int}[\varphi(t)/2\pi]}$ where $I_0 = \sqrt{D} e \Delta$ & $x_n = Q_n - P_n$

similar models have been used for Landau-Zener transitions near avoided crossings in conventional junctions
Averin et al. 1995, Pikulin et al. 2011, Sau et al. 2012 …
Realistic Circuit: Time Scales

\[
P_{2(n+k)} = P_{2n}^\infty + (1-s)^{2k} (P_{2n} - P_{2n}^\infty),
\]
\[
P_{2(n+k)+1} = P_{2n+1}^\infty + (1-s)^{2k+1} (P_{2n} - P_{2n}^\infty)
\]

with \( P_n^\infty = [1 - (-1)^n s/(2 - s)]/2 \)

- Markov process defines characteristic time scale

\[
\tau_s = -2\pi / [eV_{dc} \ln(1-s)] \approx 2\pi / (seV_{dc}) \quad \text{for } s \ll 1
\]

- Resistive environment provided by the circuit sets time scale \( \tau_R \) for adjusting the phase across the junction
• consider a voltage-biased Josephson junction in series with an external resistance

\[ V(t) = R I_S(t) + \frac{1}{2e} \dot{\phi}(t) \]

where \[ V(t) = V_{dc} + V_{ac} \cos(\Omega t) \]
\[ I_S(t) = \pm I_0 \sin(\varphi(t)/2) \]

• with \[ \varphi(t) = k\Omega t + \frac{2eV_{ac}}{\Omega} \sin(\Omega t) + \phi(t) \] for \[ 2eV_{dc} \approx k\Omega \], one finds

\[ \dot{\phi}(t) = 2eV_{dc} - k\Omega + 2eRI_0J_{-k/2} \left( \frac{eV_{ac}}{\Omega} \right) \sin \frac{\phi(t)}{2} \]
RSJ Model

\[ \dot{\phi}(t) = 2eV_{dc} - k\Omega + 2eRI_0J_{-k/2} \left( \frac{eV_{ac}}{\Omega} \right) \sin \frac{\phi(t)}{2} \]

- characteristic time scale: \( \tau_R^{(k)} = 1/(eRI_k) \) with \( I_k = I_0 |J_{-k/2}(eV_{ac}/\Omega)| \)

if \( \tau_R^{(k)} \ll \tau_s \), parity conserved during formation of step: only even Shapiro steps! even-odd effect

- shape of the step:

\[
I_{dc} = \sum_k \frac{\delta V_k}{R} \left\{ 1 - \theta \left[ 1 - \left( \frac{RI_k}{\delta V_k} \right)^2 \right] \sqrt{1 - \left( \frac{RI_k}{\delta V_k} \right)^2} \right\}
\]

where \( \delta V_k = V_{dc} - k\Omega/(2e) \) and \( k \) even
\( \tau_s \ll \tau_R : \text{Average Current} \)

- non-adiabatic processes important

average current:

- determined by long-time probabilities

\[
P_n^\infty = \frac{1}{2} \left( 1 - (-1)^n \frac{s}{2 - s} \right)
\]

independent of initial state & \( 4\pi \)-periodic

\[
\langle I(t) \rangle = \frac{s}{2 - s} I_0 \left| \sin \frac{\varphi(t)}{2} \right|
\]

- \( 2\pi \)-periodic
- strongly suppressed for \( s \ll 1 \)

NOTE: for \( V_{\text{ac}} \ll V_{\text{dc}} \) & \( \Omega \ll \delta \), dependence of \( s \) on ac bias negligible

\[ \rightarrow \text{Shapiro steps suppressed} \]
\( \tau_s \ll \tau_R : \text{Noise} \)

\[
S(\omega) = 2 \int_0^\infty d\tau \cos(\omega \tau) \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \delta I(t) \delta I(t+\tau) \rangle
\]

determined by correlator

\[
\langle I(\varphi_1)I(\varphi_2) \rangle = I_0^2 \sin \frac{\varphi_1}{2} \sin \frac{\varphi_2}{2} \{ x_{n_2}(Q_n = 1)Q_{n_1}^\infty - x_{n_2}(P_n = 1)P_{n_1}^\infty \}
\]

where \( n_i = \text{Int}[\varphi_i/2\pi] \)
finite-frequency noise: peak at $\omega = \pm eV_{dc}$

$$S_0(\omega) \approx \frac{1}{2} \frac{I_0^2}{(\omega + eV_{dc})^2 + (seV_{dc}/\pi)^2}$$

for $\omega \approx \pm eV_{dc}$ & $s \ll 1$

- peak width $\delta \omega = s\frac{eV_{dc}}{\pi} \rightarrow$ characteristic time scale $\tau_s$

similar results for finite-size Majorana nanowires

equivalently: fractional AC effect in transient regime

Pikulin & Nazarov 2011
San-José, Prada & Aguado 2012
noise peak shifted to lower (more accessible?) frequencies under microwave irradiation:

\[
S_0(\omega) \approx \frac{1}{2} I_0^2 J_k^2 \left( \frac{eV_{ac}}{\Omega} \right) \frac{seV_{dc}/\pi}{[\omega + (eV_{dc} - k\Omega)]^2 + (seV_{dc}/\pi)^2}
\]
signatures of Majorana fermions are subtle …

Non-equilibrium properties of S-TI-S junctions:

• interplay between 2 time scales:
  ➢ lifetime of the Majorana bound state $\tau_s$
  ➢ phase adjustment time $\tau_R$

• even/odd effect in Shapiro steps for $\tau_s \gg \tau_R$

• clear signatures of fractional Josephson effect in finite-frequency noise

D.M. Badiane, M. Houzet & JSM, PRL 107, 177002 (2011)
Alternative Approach: Scattering Theory
Electrons/holes gain/lose energy $eV_{dc}$ when traversing the barrier.

Only quasi-particles with energy $|\varepsilon| > \Delta$ can escape into reservoir.
Scattering Formalism
• **Andreev reflection:** $r_A(\epsilon) = a(\epsilon)$

• **scattering matrix for electrons:**

$$S_e = \begin{pmatrix} r & t \\ t & r \end{pmatrix}$$

+ **applied voltage:**

$$S_e = \begin{pmatrix} r & te^{-ieVt} \\ te^{ieVt} & r \end{pmatrix}$$

• **scattering matrix for holes:**

$$S_h = \begin{pmatrix} -r^* & t^*e^{-ieVt} \\ t^*e^{ieVt} & -r^* \end{pmatrix}$$
Scattering Formalism

\[ \Psi_{e, \rightarrow} \sim \sum_n \left( \begin{array}{c} J\delta_n,0 + a_{2n}A_n \\ A_n \\ B_n \\ a_{2n}B_n \end{array} \right) e^{-i(\epsilon + 2neV)t} \]

- derive recursion relations for coefficients \( A_n, B_n \)

- obtain expression for current:

\[ I(t) = \sum_n I_n e^{2ineVt} \]

where

\[ I_n = \frac{e}{h} \left\{ D e V \delta_{n,0} - \int d\epsilon \tanh \frac{\epsilon}{2T} \left( J(a_{2n}^*A_n^* + a_{-2n}A_{-n}) + \sum_k (1 + a_{2(n+k)}^*a_{2k})(A_{n+k}^*A_k - B_{n+k}^*B_k) \right) \right\} \]

compute \[ S(\omega) = \int d\tau \, e^{i\omega \tau} \langle \delta \hat{I}(t) \delta \hat{I}(t+\tau) + \delta \hat{I}(t+\tau) \delta \hat{I}(t) \rangle \]

\[
S(\omega) = \frac{e^2}{2\hbar} \sum_{\pm \omega} \sum_p \int d\epsilon d\epsilon' \, J(\epsilon) J'\left\{ (f(\epsilon)[1 - f(\epsilon')] + f(\epsilon')[1 - f(\epsilon)]) \times \right.

\[
\left. \times \left[ \delta(\epsilon - \epsilon' \pm \omega + 2p eV) \left[ \sum_n \left( A^*_{n+p} A'_{n} + a^*_{2(n+p)} a'^*_{2n} \tilde{A}^*_{n+p} \tilde{A}'_n - (1 + a^*_{2(n+p)} a'^*_{2n}) B^*_{n+p} B'_n \right)^2 
\right.
\right.
\]

\[
+ \sum_n \left( 1 + a^*_{2(n+p)} a'^*_{2n+1} \right) (C^*_{n+p} C'_n - D^*_{n+p} D'_n)^2 \right]
\]

\[
+ 2\delta(\epsilon - \epsilon' \pm \omega + (2p - 1)eV) \left[ \sum_n \left( A^*_{n+p} C'_{n} + a^*_{2(n+p)} a'^*_{2n} \tilde{A}^*_{n+p} C'_n - (1 + a^*_{2(n+p)} a'^*_{2n+1}) B^*_{n+p} D'_n \right)^2 \right]
\]

\[
+ (f(\epsilon)f(\epsilon') + [1 - f(\epsilon)][1 - f(\epsilon')]) \times \left.
\left. \times \left[ \delta(\epsilon + \epsilon' \pm \omega + 2p eV) \sum_n \left[ A'_{p-n} B'_n - B_{p-n} A'_n - a_{2(p-n)} a'^*_{2n} (\tilde{A}'_{p-n} B'_n - B_{p-n} \tilde{A}'_n) \right)^2 \right.
\right.
\]

\[
+ \delta(\epsilon + \epsilon' \pm \omega + (2p + 1)eV) \left[ \sum_n \left( 1 - a_{2(p-n)} a'^*_{2n+1} \right) (D_{p-n} C'_n - C_{p-n} D'_n)^2 \right] \right.
\]

\[
+ 2\delta(\epsilon + \epsilon' \pm \omega + (2p + 1)eV) \left[ \sum_n \left[ A'_{p-n} D'_n - a_{2(p-n)} a'^*_{2n+1} \tilde{A}'_{p-n} D'_n - (1 - a_{2(p-n)} a'^*_{2n+1}) B_{p-n} C'_n \right)^2 \right] \right\}.
\]
Results: Noise

![Graph showing frequency dependent current noise.]

peak at $\omega = \pm eV_{dc}$

$\rightarrow$ signature of the fractional Josephson effect

$I_\Delta = \frac{eD\Delta}{2\pi}$
Comparison: Effective Model vs Numerical Results

effective model in the limit $R \ll 1$ and $\tau_R \to \infty$:

$$S(\omega) = \frac{4sI_j^2}{\pi(2-s)} \frac{(eV_{dc})^3}{[\omega^2-(eV_{dc})^2]^2} \frac{4\cos^2 \frac{\pi \omega}{2eV_{dc}}}{4\cos^2 \frac{\pi \omega}{2eV_{dc}} + \frac{s^2}{1-s}}$$

• fit numerical results with $s$ as a free parameter

$D = 0.9$ & $eV_{dc} = 0.025\Delta$

$D = 0.9$ & $eV_{dc} = 0.05\Delta$
Comparison: Effective Model vs Numerical Results

compare with $s(\lambda)$ (Demkov-Osherov):

$$\lambda = R^{3/2} \Delta / (eV_{dc})$$
Additional Signatures: DC Current

Transparent junction \((D = 1)\): \(I_{S-TI-S} = \frac{1}{2} I_{S-N-S}\)

Tunnel junction \((D \rightarrow 0)\): current onset at \(eV_{dc} = \Delta\)

For comparison:

S-N-S

\(G_N = \frac{2e^2}{h}\)

\[G_N = \frac{2e^2}{h}\]

\[G_N = \frac{2e^2}{h}\]

\[G_N = \frac{2e^2}{h}\]
Additional Signatures: DC Current

- subgap MAR structures at \( neV_{dc} = \Delta \)  
  (vs \( neV_{dc} = 2\Delta \) for S-N-S)

\[
\begin{align*}
G = \frac{e^2}{h} \\
n = 2 \\
n = 3
\end{align*}
\]
signatures of Majorana fermions are subtle …

Non-equilibrium properties of S-TI-S junctions:

• even/odd asymmetry in Shapiro steps for $\tau_s \ll \tau_R$

• clear signatures of fractional Josephson effect in finite-frequency noise

• DC current
  – subgap structures due to multiple Andreev reflections at $eV_{dc} = \Delta/n$
  – tunneling limit: current onset at $eV_{dc} = \Delta$

D.M. Badiane, M. Houzet & JSM, PRL 107, 177002 (2011)
Thank you!