UNIVERSAL NEGATIVE MAGNETORESISTANCE IN THE VARIABLE RANGE HOPPING REGIME: A CONSEQUENCE OF THE SIGN DISORDER OF THE ELECTRON WAVE FUNCTION.

Boris Spivak and L.I.
1. Introduction: Magnetoresistance as a consequence of interference in weakly and strongly disordered conductors
   a. Solution of directed polymer problem in two opposite limits, RSB and dominance by a single path.
   b. Conclusions for Statistical properties of electron tunneling amplitude (polymer free energy)
3. Negative scattering amplitudes lead to universal magnetic field dependence of magnetoresistance.
6. Application to physical systems and comparison with data.
7. Conclusions.
MAGNETORESISTANCE OF DISORDERED CONDUCTORS
\[ \frac{\delta R}{R} \sim (\omega_H \tau)^2 \]  because the scattering is characterized by only one time the only dimensionless parameter is $\omega_H \tau$ and magnetic field breaks time invariance, so linear terms are impossible

Magnetic field bends the trajectories → magnetoresistance is small and positive
The total probability to get from A to B:

\[ w_{AB} = \left| \sum_i A_i \right|^2 = \sum_i |A_i|^2 + \sum_{ij} |A_i A_j| \]

The phase gain:

\[ \Delta \varphi = \frac{1}{\hbar} \int_A^B \vec{p} d\vec{l} \gg 1 \quad p - \text{the Fermi momentum} \]

The interference term vanishes

\[ \left\langle \sum_{ij} A_i A_j \right\rangle = 0 \]

due to the disorder averaging, but NOT for the time-reversed paths (loops)

The self-intersecting (loop-like) trajectories:

\[ \vec{p} \Rightarrow -\vec{p} \quad \vec{l} \Rightarrow -\vec{l} \quad \Delta \varphi = 0 \]

\[ w_0 = |A_1|^2 + |A_2|^2 + 2 \text{Re } A_1 A_2 = 4|A_1|^2 \]

As a result, the total scattering probability at site O increases, and the conductance decreases. The corresponding corrections to the transport properties are known as the weak localization corrections.
WEAK LOCALIZATION: NEGATIVE MAGNETORESISTANCE

Classically weak magnetic fields: no trajectory “bending”.

Aharonov-Bohm phase acquired by the loops:

\[ \vec{p} \Rightarrow \vec{p} - \frac{e}{c} \vec{A} \quad \Psi \Rightarrow \Psi \exp \left( i \frac{e}{\hbar c} \oint \vec{A} d\vec{l} \right) \]

\[ \Delta \varphi_H = \frac{2e}{\hbar c} \oint \vec{A} d\vec{l} = \frac{2e}{\hbar c} \oint \vec{H} d\vec{s} = 2\pi \frac{\Phi}{\Phi_0} \]

- the phase difference between CW and counter-CW loops increases with field.

\[ \Phi_0 = \frac{\hbar c}{2e} = 2.07 \cdot 10^{-7} \text{ G} \cdot \text{cm}^2 \]

- the flux quantum (~20G in 1μm²)

\[ \omega_0 = |A_1|^2 + |A_2|^2 + 2 |A_1| |A_2| \cos(\Delta \varphi_H) = 2A^2 \left[ 1 + \cos(\Delta \varphi_H) \right] \]

\[ L_H = \sqrt{\frac{\Phi_0}{2\pi H}} \]

- the magnetic length (the characteristic length at which two interfering waves propagating along a loop in opposite directions acquire the phase difference ~ 2π)
WEAK LOCALIZATION: OBSERVATION OF NEGATIVE MAGNETORESISTANCE

\[
\frac{R(H) - R(0)}{R(0)} \leq \frac{2\pi^2 \hbar}{R_{sq} e^2} \left( \frac{2\pi^2 \hbar}{R_{sq} e^2} \right) \ll 1
\]

**Weak Localization magnetoresistance:**
- negative
- anisotropic in 1D and 2D (purely orbital effect)
- observed in classically weak magnetic fields
- Small away from MI transition

Misha Gershenson *et al.*, ‘81
LARGE NEGATIVE MAGNETORESISTANCE

\[ \delta \sigma(H) / \sigma_0 \sim 1 \]

Non-analytical \( \sigma(H) \)?

\[ \sigma(T) \propto \exp\left[ -(T_M / T)^{1/3} \right] \]

InO films, Milliken and Ovadyahu (1990)
Interference in weak localization: loops are important

Interference in strongly localized regime: no return paths.
Tunneling amplitude through a given path:

\[ A_{\text{path}} = \prod_k G_{k,k+1} \frac{b_k}{\epsilon - \epsilon_k} \quad G_{k,k+1} = \frac{1}{r_{k,k+1}^{1/2}} \exp(-r_{k,k+1} / \xi) \]

Summing over all paths:

\[ A_{ij} = \sum_{\text{path}} \prod_k G_{k,k+1} \frac{b_k}{\epsilon - \epsilon_k} \]
Stability of ground state with respect to one particle hop:

The density of states at the Fermi level must vanish at \( T = 0 \).

Self-consistent argument for the density of states:

\[
R_E = \frac{e^2}{E} \quad ; \quad R_E^D \cdot \int_0^E \rho(E) \, dE \leq 1
\]

Parabolic pseudogap in \( D = 3 \).
Linear density of states in \( D = 2 \) and 3D Coulomb.
Exponential suppression in \( D = 2 \) and 2D Coulomb

Effect of many particle hops?
→ formation of glassy state
→ modify the numerical coefficient \( \alpha \)
Each site is characterized by random scattering amplitude $\mu$

Wave function propagation from site to site is constant $u$

$$\mu_a = \frac{1}{E - \xi_a}$$

What is distribution of on site energies?

At very low temperatures:
- Coulomb gap

If Coulomb interaction is small
- Constant

For very large (2D)
- pseudogap
STATISTICAL PROPERTIES OF AMPLITUDES

\[ Z = \sum_{\text{path}} \left\{ \prod_k G_{k,k+1} \frac{b_k}{\epsilon - \epsilon_k} \right\} \]

Very different statistical properties depending on the distribution of \( \mathcal{P}(\epsilon - \epsilon_k) \)

**Result:** in most cases the main contribution comes from one path (RSB)

Simplified problem 1:

A. \( G_{k,k+1} \rightarrow G \)

B. \( \epsilon - \epsilon_k > 0 \)

C. No correlation between pathes (Bethe lattice)

Simplified problem 2: Small scattering amplitudes \( \frac{b_k}{\epsilon - \epsilon_k} \ll 1 \) in two dimensions.

Quantitative results are very different but the dominance of a single path persists.
EQUIVALENCE OF HOPPING AND DIRECTED POLYMER PROBLEMS; STATISTICAL PROPERTIES
PARTITION FUNCTION OF DIRECTED POLYMER.

\[ Z = \sum_{\{i[k]\}} \prod_k \frac{g}{K} \frac{\text{Tanh}[\xi_{k,i[k]} / T]}{\xi_{k,i[k]}} \]  \text{Tanh}[\xi_{k,i[k]} / T] \text{ introduced for regularization}

T is not the physical temperature for hopping problem

Non-trivial physics is due to the fact that \( Z \) is not necessarily self-averaging quantity!
Typical \( Z_{\text{typ}} = \exp N<f> \) might be different from \( <Z> \).
To find average \( <f> \) use replica trick:

\[ Z^n = \sum_{\{i_a[k]\}} \prod_k \frac{g}{K} \frac{\text{Tanh}[\xi_{k,i_a[k]} / T]}{\xi_{k,i_a[k]}} \quad a = 1..n \]

Solve the problem for \( n \) replicas and continue to \( n=0 \). Similar problems were solved in the context of directed polymer physics \((Derrida and Spohn)\).
At low \( T \) replica symmetry breaks down.
ONE STEP RSB

\[ Z^n = \sum_{\{i_a[k]\}} \prod_k \frac{g}{K} \frac{\text{Tanh}[\xi_{k,i_a[k]} / T]}{\xi_{k,i_a[k]}} \quad a = 1..n \]

Below transition paths are grouped into bundles of m path in one bundle

Bundled average (n/m bundles):

\[ Z^n = \left( K^{n/m} \left( \frac{g}{K} \right)^n \int_0^\infty \left[ \frac{\text{Tanh}(\xi / T)}{\xi} \right]^m d\xi \right)^N \]
Bundled average (n/m bundles):

\[ Z^n = \left( K^{n/m} \left( \frac{g}{K} \right)^n \int_0^\infty \left[ \frac{\tanh(\xi / T)}{\xi} \right]^m d\xi \right)^N \]

\[ f = \ln \frac{g}{K} + \frac{1}{m} \left[ \ln K + \ln \int_0^\infty \left[ \frac{\tanh(\xi / T)}{\xi} \right]^m d\xi \right] \]

Before continuation to \( n \to 0 \) \( n > m > 1 \), after \( n < m < 1 \)

RSB occurs when \( m^* \) minimizing \( f(m) \) becomes \( m < 1 \).

\[ T \to 0 \quad m^* : \quad \ln \frac{K}{1-m^*} = \frac{m^*}{1-m^*} \quad \text{always has a solution} \]

Always RSB for distributions with small gaps
Effective number of paths (analogue of participation ratio):

\[
\chi = \frac{\sum \prod_{k} \left[ \frac{\tanh[\frac{\xi_{k,i[k]} / T}{\xi_{k,i[k]}}]}{\xi_{k,i[k]}} \right]^2}{\sum \prod_{k} \left[ \frac{\tanh[\frac{\xi_{k,i[k]} / T}{\xi_{k,i[k]}}]}{\xi_{k,i[k]}} \right]^2}
\]

If M paths contribute to Z roughly equally, \( Z/M \rightarrow \chi = \frac{1}{M} \)

Compute \( \langle \log(\chi) \rangle \) by replica symmetry breaking in numerator and denominator separately:

\[
f_1 = \frac{1}{m} \left[ \ln K + \ln \int_0^{\infty} \left[ \frac{\tanh(\xi / T)}{\xi} \right]^m d\xi \right]
\]

\[
f_2 = \frac{1}{m} \left[ \ln K + \ln \int_0^{\infty} \left[ \frac{\tanh(\xi / T)}{\xi} \right]^{2m} d\xi \right]
\]

\[
m_2^* = m^* / 2 \rightarrow f_2(m_2^*) = 2f(m^*)
\]

\( \log(\chi) = 0 + O(1) \)

Conclusion: only a small number of path contribute
Linearized equation

\[ h_{k+1} = \frac{g}{K} \sum_j h_{k,j} \tanh \frac{\xi_{k,j}}{T} \]

Recursion for probability

\[ P(h) = \prod dh_i P(h_i) \delta \left[ h - \frac{g}{K} \sum_i h_{k,i} \tanh \left( \frac{\xi_{k,i}}{T} \right) \right] \]

Laplace transform

\[ P(s) = \left[ \int_0^1 d\xi P \left( s \frac{g \tanh \frac{\xi}{T}}{\xi} \right) \right]^K \]

Look for the solution \( P(s) = 1 - As^x \)

\( x = 1 \) corresponds to RS phase \( A = \langle h \rangle \),

\( x < 1 \) to RSB in which case

This is exactly one of RSB equations

\( f(x) = 0 \)

Minimum of \( f(x) \) – most divergent solution

Second RSB equation

\[ 1 = K \int_0^1 \frac{d\xi}{\xi} \left( \frac{g \tanh \frac{\xi}{T}}{K \xi} \right)^x \]

\[ \frac{d}{dx} \int_0^1 \frac{d\xi}{\xi} \left( \frac{g \tanh \frac{\xi}{T}}{K \xi} \right)^x = 0 \]
Small fluctuations of scattering amplitudes $\mu_a = 1 + \zeta_a$

$$\Psi_n = \sum_{x(k)} \prod_{k} (1 + \zeta_{x(k)})$$

$\Psi_n \leftrightarrow Z$ - partition function

$F = -\ln Z$ - free energy of polymer

$$\frac{dZ}{dz} = D \nabla^2 Z + \zeta(z, x) Z$$ - continuum limit

- diffusion in random media

What cannot be mapped:
- Random signs of scattering
- Effect of magnetic field
In continuum limit averaging \( n \) copies of the system:

\[
\langle Z^N \rangle = \exp(-\int H d\tau)
\]

\[
H[y(x)] = \int_{-L}^{0} dx \left\{ \frac{c}{2} \sum_{a=1}^{n} \left[ \partial_x y_a(x) \right]^2 - \frac{U_0}{2T} \sum_{a,b=1}^{n} \delta[y_a(x) - y_b(x)] \right\}
\]

Solution for the ground state: \( \psi_0(y) = \exp\left(-\kappa \sum_{a,b} |y_a - y_b| \right) \quad \kappa = \frac{cU_0}{4T^3} \)

Partition function of \( n \) copies \( Z_N = \exp \left[ \frac{cU_0^2 N(N^2 - 1)L}{24T^4} \right] \)

\[
Z_N = \int dF \mathcal{P}(F) \exp(NF)
\]

Third moment of free energy: \( \langle F^3 \rangle = \frac{cU_0^2 L}{4T^4} \)

Typical free energy \( F_{\text{typ}} \sim L^{1/3} \quad u(L) \sim L^{2/3} \) - transverse deviations

Contribution of different paths scale as \( \exp(-u^{1/2}) \sim \exp(-L^{1/3}) \)
Solution for the ground state: \( \psi_0(y) = \exp\left(-\kappa \sum_{a,b} |y_a - y_b| \right) \quad \kappa = cU_0 / 4T^3 \)

Third moment of free energy: \( \left\langle F^3 \right\rangle = cU_0^2 L / 4T^4 \)

Second moment of free energy: \( \left\langle F^2 \right\rangle = 0 \)

Impossible for any positive P(F)!

Root of the problem is neglect of excited states for the polymer of finite length L and approximation

\[ F = E_0 L \]

Typical free energy \( F_{typ} \sim L^{1/3} \quad u(L) \sim L^{2/3} \) - transverse deviations

Contribution of different paths scale as \( \exp(-u^{1/2}) \sim \exp(-L^{1/3}) \)
Consider two polymer configurations, one ending in point \( u \), another at point \(-u\) and replicated it:

\[
Z_r(n, m; L, u) \equiv \langle Z^n(L, u; V)Z^m(L, -u; V) \rangle_v
\]

\[
= \left\langle e^{\frac{-nF^+}{T}} e^{\frac{-mF^-}{T}} \right\rangle_v = \left\langle e^{\frac{-(n+m)\overline{F}}{T}} e^{\frac{-(n-m)F'}{2T}} \right\rangle_v
\]

\[
= \int_{-\infty}^{+\infty} d\overline{F}dF' P_{L,u}(F', \overline{F}) e^{-\frac{(n+m)\overline{F}}{T}} e^{-\frac{(n-m)F'}{2T}}
\]

\[
H[y(x)] = \int_{-L}^{0} dx \left\{ \frac{c}{2} \sum_{a=1}^{n} [\partial_x y_a(x)]^2 - \frac{U_0}{2T} \sum_{a,b=1}^{n} \delta[y_a(x) - y_b(x)] \right\}
\]

Describes \( n+m \) particles interacting via repulsion \( U \)

Ground state:

\[
\psi_0(y) = \exp\left( -\kappa \sum_{a,b} |y_a - y_b| \right) \quad E_0(n + m) = -cU_0^2(n + m)[(n + m)^2 - 1] / 24T^4
\]

(Dotsenko, Geshkenbein, LI)
\[ \psi_0(u) = \exp \left( -\kappa \sum_{a,b} |y_a - y_b| \right) = \exp \left( -4\kappa |u| nm \right) \]

\[ Z_r(n,m;L,u) = 2^{n+m-1} e^{-E_0(n+m)L/T} e^{-4\kappa |u| nm} \]

\[ 4\kappa |u| nm = \kappa |u| \left[ (n+m)^2 + (n-m)^2 \right] \]

\[ Z_r^-(n-m;u) = \int dF' P_u(F') e^{-(n-m)F'/2T} \]

\[ P_u(F') = \left( \frac{T}{4\pi c U_0 |u|} \right)^{1/2} \exp \left( -\frac{TF'^2}{4c U_0 |u|} \right) \]

\[ \begin{aligned} F' &\sim u^{1/2} \\ u^2 / L &\sim F' \end{aligned} \Rightarrow u \sim L^{2/3}, \quad F' \sim L^{1/3} \]

\[ \mathcal{P}_L(F) \quad \text{For the finite length the result is much more complicated:} \]

\[ \text{Tracy-Widom distribution} \quad \text{(Dotsenko, Calabrese, Le Doussal)} \]
NEGATIVE SCATTERING AMPLITUDES AND CONSEQUENCES FOR MAGNETORESISTANCE
\[ A_{path} = \prod_k G_{k,k+1} \frac{b_k}{\epsilon - \epsilon_k} \quad \text{where} \quad G_{k,k+1} = \frac{1}{r_{k,k+1}^{1/2}} \exp(-r_{k,k+1} / \xi) \]

Can be mapped onto directed polymer partition function only of all \( \epsilon - \epsilon_k > 0 \) because \( Z > 0 \) in any thermodynamic problem.

What is effect of \( \epsilon - \epsilon_k < 0 \)?

1. Does it make \( Z \) random in sign?
2. Does it change the statistics of \( Z \)?

For a directed polymer (dense impurities) \( Z \) is controlled by a single path \( \rightarrow \) just one negative amplitude is sufficient to change sign \( Z \).

Sign \( Z \) becomes random at \( \ln_{\text{neg}} \sim 1 \)

Statistics of \( Z \) does not change by negative amplitudes because it is dominated by a single path \( \rightarrow \) weak interference effects
HOPPING REGIMES: RARE SCATTERERS

Plane scattering by impurities:

\[
\frac{\mu_a}{r - r_a} \exp\left(-\frac{|r - r_a|}{\xi}\right)
\]

\[
\exp\left(-\frac{rn}{\xi}\right) + \frac{\mu_a}{r - r_a} \exp\left(-\frac{|r - r_a|}{\xi}\right)
\]

\(\mu\) - scattering amplitude

Areas of negative wave function do not overlap if \(Sn < 1\).

\(S = \mu^{3/2}\)

Conclusion: for small \(n\) the scattering are largely irrelevant.
HOPPING REGIMES: RARE SCATTERERS

Areas of affected wave function do not overlap if $Sn < 1$.

$S = \mu^{3/2}$

Areas of affected wave function overlap if $Sn > 1$.

$S = \mu^{3/2}$

Conclusion:
(Shklovski – Spivak)
For negative scattering amplitude there is transition between sign ordered and disordered phases.
DIRECTED POLYMERS: QUALITATIVE PICTURE

- Propagation from A to B is dominated by a ‘single’ path, i.e. a small number of paths in a narrow tube.
- Typical fluctuations in the free energy grow as \( L^{1/3} \)
- Typical deviation grows as \( L^{2/3} \)
- Distribution of free energy differences at distance X is Gaussian
- Distribution of free energy at the end is Tracy-Widom.

These conclusions are exact for weak scattering for which mapping to continuous problem is exact.

Is it correct for strong (physical) scattering? Check by numerics.

Strong and weak scattering on random graph belong to different universality classes. Strong in 2D might lead to phase transition between sign ordered and disordered phases.
Propagation from A to B is dominated by a ‘single’ path, i.e. a small number of paths in a narrow tube.

A typical alternative path has a free energy that is larger than that of the dominant path and grows as $L^{1/3}$ so typically there is no interference.

Effect of magnetic field happens only if two free energies coincide which happens with probability $L^{-1/3}$.

Interference is of the order of unity if phases differ by $\sim \pi$ so that flux $B \times L \sim \Phi_0$. Transverse $X \sim L^{2/3}$ so this condition is satisfied at $B L^{5/3} \sim \Phi_0$.

The interference decreases the amplitude by $O(1)$, which decrease the localization length by $1/L$ with probability $1/L^{1/3}$.

$$\delta \xi \propto -L^{-4/3} \propto -B^{-4/5}$$ non analytical and universal correction that translates into

$$R(H) = \exp \left[ \left( \frac{T_0}{T} \right)^a \left[ 1 + \frac{\delta \xi}{\xi_0} (Ba^2)^{4/5} \right] \right]$$
EFFECT OF MAGNETIC FIELD IN CASE OF NEGATIVE SCATTERING AMPLITUDES

Different paths interfere even without magnetic field due to random signs:

\[ \Psi_{A \rightarrow B} = \Psi_1 \pm \Psi_2 \quad \text{in the presence of magnetic field is replaced by} \]

\[ \Psi_{A \rightarrow B} = \Psi_1 + e^{i\phi} \Psi_2 \quad \text{with random phase } \phi \]

The amplitude gets larger, so effect of localization length is negative.

The effect is large if \( F_1 - F_2 \sim O(1) \) which happens with \( P \sim 1/L^{1/3} \)

\[ \delta \xi \propto L^{-4/3} \propto B^{-4/3} \quad \text{non analytical and universal correction} \]

that translates into \( R(H) = \exp \left( \left( \frac{T_0}{T} \right)^a \left[ 1 - \frac{\delta \xi}{\xi_0} (Ba^2)^{4/5} \right] \right) \)
NUMERICAL EVIDENCE FOR UNIVERSALITY FOR ALL TYPES OF SCATTERINGS
Free energy fluctuations scale as $L^{0.345}$
Assume that $0.345 = 1/3$ (?)
CHECK B-DEPENDENCE OF LOCALIZATION LENGTH

Gapped density of states
Expected ‘universal’ behavior at all fields

\[ \mu_a = \frac{1}{\xi_a} \quad P(1 > \xi > 1/2) = 2 \quad \text{- gapped} \]

\[ \mu_a = \frac{1}{\xi_a} \quad P(\xi > 0) = 2\xi \quad \text{- linear} \]

Linear density of states
Expected ‘universal’ behavior at very low fields, surprisingly larger fields look ‘universal’ (power law) but with a wrong exponent
EFFECT OF NEGATIVE SCATTERERS ON POLYMER FREE ENERGY

\[ \Psi_n = \sum_{x(k)} \prod_k \frac{1}{\xi_{x(k)}} \] - serious problem

\[ \Psi_n \leftrightarrow Z \] - no longer correct if some \( \xi_x < 0 \)

\[ F = -\ln Z \] - free energy is no longer real

What happens to scaling behavior?

Free energy fluctuations scale as \( L^{0.33} \)

Higher moments scale as Tracy-Widom distribution
CHECK B-DEPENDENCE OF LOCALIZATION LENGTH IN CASE OF NEGATIVE SCATTERERS

Gapped density of states
Expected ‘universal’ behavior at small fields

\[ \mu_a = \frac{1}{\xi_a} \quad P(1 > |\xi| > 1/2) = 1 \quad - \text{gapped} \]

Equally probably positive and negative

\[ \mu_a = \frac{1}{\xi_a} \quad P(\xi) = |\xi| \quad |\xi| < 1 \quad - \text{linear} \]

Linear density of states
Expected ‘universal’ behavior at very low fields.

At larger fields look ‘universal’ (power law) but with a wrong exponent
Rare scatterers → phase transitions between random sign and sign ordered phase (Shklovski – Spivak). Does it survive in strong scattering limit?

Sign transition for a gapfull distribution
Rare scatterers → phase transitions between random sign and sign ordered phase (Shklovski – Spivak). Does it survive in strong scattering limit?

If wave function is determined by a single path
Any concentration of negative scatterings is sufficient to make sign completely random
$L_c P_\sim 1$

Conjecture: for linear and uniform DOS the sign is always random at large scales

$P_\sim 10^{-4}$
WHAT IS MAGNETORESISTANCE FOR SMALL P?

Expect universal negative at small B and positive at large B.

Linear density of states, ‘universal’ dependence of R(H).
Strange exponent: \( B_0 \sim n^{2.8} \) instead of \( B_0 \sim n^{1.6} \).
APPLICATION TO EXPERIMENTAL SYSTEMS
**HOPPING CONDUCTIVITY**

Hopping: Optimize

\[ e^{-2 R_{ij} / \xi} \cdot e^{-\Delta E_{i \rightarrow j} / kT} \]

**Mott conductivity**:

Constant density of states

\[ \rho(E = 0) = N_0 \]

\[ \sigma(T) \propto \exp\left[-\left(\frac{T_M}{T}\right)^{1/d+1}\right] \]

\[ T_M = 18/\xi_{loc}^3 N_0 \approx 1500K \]

**Efros-Shklovskii conductivity**:

Coulomb gap

\[ \rho(E) = \alpha \left(\frac{\kappa}{e^2}\right)^3 E^{d-1} \]

\[ \sigma(T) \propto \exp\left[-\left(\frac{T_{ES}}{T}\right)^{1/2}\right] \]

\[ T_{ES} = 2.8 e^2 / \xi_{loc} \kappa \approx 12K \]

---

Electron hopping rate is given only approximately by
\[ e^{-2R_{ij}/\xi} \cdot e^{-\Delta E_{i\rightarrow j}/kT} \]
More precisely one should compute the phonon matrix element:
\[
\Gamma_{ij} = W_{ij} \cdot e^{-\Delta E_{i\rightarrow j}/kT}
\]
\[
W_{ij} = \int dq |M(q)|^2 \delta(\varepsilon_i - \varepsilon_j - uq)
\]
\[
M(q) = \int dr \Psi_i(r) \Psi_j(r) \exp(iqr)
\]

Dependence of magnetic field can be approximated by
\[
M(q, B) \sim A_{ij}(B) \sim \exp \left[ \frac{-r_{ij}}{\xi_0 + \delta\xi(B)} \right]
\]
Provided that change is large:
\[
\frac{\delta\xi(B)r_{ij}}{\xi_0^2} \gg 1
\]

Experimentally it is hardly possible to measure resistances larger than \(10^9 - 10^{10} \Omega\)
So \(r/\xi \lesssim 10 - 15\)

What happens when \(\frac{\delta\xi(B)r_{ij}}{\xi_0^2} < 1\) ?
TUNNELING AMPLITUDE AT SHORT SCALES

At large fields

\[ M(q, B) \sim A_y(B) \sim \exp \left[ -r_{ij} \frac{\delta \xi(B)}{\xi_0^2} \right] \sim \exp B^{4/5} \]

What happens for smaller fields?
Consider two interfering paths. Amplitude acquires non-analyticity due to zeros of wave function:

\[ \langle \ln \left| \frac{A(B)}{A(0)} \right| \rangle = \int dA_1 dA_2 \ln |A_1 - A_2 \exp(i\phi_B)| \sim |\phi_B| \]

This non-analyticity disappears in which has a sum over different \( q \)

So that zero of \( M \) for one direction of \( q \) does not coincide with another zero.

However the main contribution comes from \( q \sim r \).

Conclusion: matrix element should show a narrow regime of linear in \( B \) dependence
Combining all different regimes together:

\[ \ln \frac{\rho(0)}{\rho(B)} = \ln \frac{W_i(B)}{W_i(0)} \sim \begin{cases} 
(B / B_0)^\alpha \gtrsim 1 & B > B_0 \\
|B| / B_0 \lesssim 1 & B_0 > B > B_* \\
B^2 / (B_*B_0) \ll 1 & B < B_* 
\end{cases} \]

\[ B_0 = \Phi_0 / r^{5/3} \xi^{1/3} \]
\[ B_* = \Phi_0 / r^2 \]

Region of linear dependence is very small in realistic situation
Numerically, the regime of ‘universal’ power dependence extends to relatively low fields
DIRECT NUMERICS FOR THE MATRIX ELEMENT
LARGE NEGATIVE MAGNETORESISTANCE OF INO

\[ \frac{\delta \sigma(H)}{\sigma_0} \sim 1 \]

Non-analytical \( \sigma(H) \)?

\[ \sigma(T) \propto \exp\left[-\left(\frac{T_M}{T}\right)^{1/3}\right] \]

InO films, *Milliken and Ovadyahu* (1990)
MAGNETORESISTANCE IN MODERATELY LOCALIZED SAMPLES

\[ \frac{R(0)}{R_0} \approx 10^3 - 10^4 \]

Ge doped GaAs films. Strongly anisotropic magnetoresistance in Mott regime Savchenko et al (1987)
And its fit to the matrix dependence. The quadratic regime is invisible in either data or theory
Indication of anomalous power dependence

Jiang et al (1992)

GaAs/Al$_x$Ga$_{1-x}$As heterostructures
Large magnetoresistance $R(0)/R(B) \sim 7$
$\delta R(B) \sim B^{1/2}$
$R(T) \sim \exp(1/T^{1/3})$ – Mott law

Pseudouniversal behavior?
\( \sigma(T) \propto \exp\left[ -\left( \frac{T_M}{T} \right)^{1/3} \right] \)

\( \frac{\delta \sigma(H)}{\sigma_0} \gg 1 \)

F doped grapehe,
Hong, Cheng, Herding, Zhu (1996)
HUGE MAGNETORESISTANCE OF MOSFET

Kravchenko et al 1998

FIG. 3. Resistivity of sample B as a function of $H_{\perp}$ in the absence of parallel magnetic field (lower curve) and in the presence of $H_{||}=34$ kOe (upper curve). $T=0.36$ K and $n_s=1.0 \times 10^{11}$ cm$^{-2}$. The inset shows $\rho(H_{\perp})$ for a low-mobility sample C, $T=0.36$ K and $n_s=1.52 \times 10^{11}$ cm$^{-2}$. 
HUGE NEGATIVE MAGNETORESISTANCE OF GE FILMS

\[ \frac{\delta \sigma(H)}{\sigma_0} \gg 1 \]

\[ \sigma(T) \propto \exp\left[-\left(\frac{T_{ES}}{T}\right)^{1/2}\right] \]

Ge films, *Mitin, Dugaev & Ihas (2007)*
CONCLUSIONS:

• Negative scattering amplitudes lead to a complete random signs of the electron wave functions are large scales.
• This results in a large negative magnetoresistance in hopping regime
• Asymptotically, the magnetoresistance should follow universal power law with exponent 4/5.
• In the intermediate regime the magnetoresistance follows an intermediate regime with exponent 0.6.
• Many data show large negative magnetoresistance in Mott regime that roughly follows power law 0.5.