Simularity Renormalization Group with Chiral Hamiltonians: Techniques & New Directions

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Introduction

**Hamilton Matrix**
pre-diagonalization (SRG) and basis transformations

**QCD-based interaction**
realistic NN+3N+4N interactions ($\chi$EFT)

**many-body problem**
ab-initio methods (IT-NCSM, CC, RGM, ...)

**nuclear structure**
confront predictions with experiment
# New Directions

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Chiral NN+3N Interactions

**standard Interaction:**
- NN $N^3$LO: Entem&Machleidt, 500 MeV cutoff
- 3N $N^2$LO: Navrátil, local, 500 MeV cutoff, fitted to Triton

**standard Interaction with modified 3N:**
- NN $N^3$LO: Entem&Machleidt, 500 MeV cutoff
- 3N $N^2$LO: Navrátil, local, with modified LECs and cutoffs, fitted to $^4$He

Next Generation Interactions

**consistent $N^2$LO Interaction:**
- NN $N^2$LO: Epelbaum et al., 450, . . . , 600 MeV cutoff
- 3N $N^2$LO: Epelbaum et al., 450, . . . , 600 MeV cutoff, nonlocal

**consistent $N^3$LO Interaction:**
- coming soon...
Similarity Renormalization Group in Three-Body Space

Roth, Neff, Feldmeier — Prog. Part. Nucl. Phys. 65, 50 (2010)
accelerate convergence by pre-diagonalizing the Hamiltonian with respect to the many-body basis

- continuous unitary transformation of the Hamiltonian

$$\tilde{H}_\alpha = U^\dagger_\alpha H U_\alpha$$

- leads to evolution equation

$$\frac{d}{d\alpha} \tilde{H}_\alpha = [\eta_\alpha, \tilde{H}_\alpha] \quad \text{with} \quad \eta_\alpha = -U^\dagger_\alpha \frac{dU_\alpha}{d\alpha} = -\eta^\dagger_\alpha$$

initial value problem with $$\tilde{H}_{\alpha=0} = H$$

- choose dynamic generator

$$\eta_\alpha = (2\mu)^2 [T_{\text{int}}, \tilde{H}_\alpha]$$

don’t get confused:

$$\alpha = \frac{1}{\lambda^4}$$

advantages of SRG: simplicity and flexibility
Three-Body Jacobi Basis

- “relative coordinates” for 3-body system

\[
\begin{align*}
\xi_0 &= \sqrt{\frac{1}{3}} [\vec{r}_a + \vec{r}_b + \vec{r}_c] \\
\xi_1 &= \sqrt{\frac{1}{2}} [\vec{r}_a - \vec{r}_b] \\
\xi_2 &= \sqrt{\frac{2}{3}} \left[ \frac{1}{2}(\vec{r}_a + \vec{r}_b) - \vec{r}_c \right]
\end{align*}
\]

- harmonic-oscillator (HO) Jacobi basis
  - antisymmetric under 1 ↔ 2:

\[
|\alpha\rangle = |[(N_1 L_1, S_1)J_1, (N_2 L_2, S_2)J_2]JM, (T_1, T_2)TM_T\rangle
\]

  - completely antisymmetric:

\[
|EijJM_JTM_T\rangle = \sum_{\alpha} c_{\alpha,i} |\alpha\rangle
\]

\[
c_{\alpha,i} = \langle EijJM_JTM_T|\alpha\rangle
\]

coefficients of fractional parentage (CFPs) by P. Navrátil
SRG in HO Jacobi Basis

- no center of mass part
  - sizable reduction of model space dimension

- coupling considers properties of interaction
  - can evolve every **TJP-channel separately**

- discrete basis enables use of CFPs
  - antisymmetrization **simple**
  - explicit consideration of the antisymmetry **decreases memory needs**

- **optimized implementation**
  - largest channel \((T = 1/2, \ J^\pi = 5/2^+)\) in **4 hours on a single node**
SRG Evolution in Three-Body Space

3B-Jacobi HO matrix elements

\[ \alpha = 0.16 \text{ fm}^4 \]
\[ \lambda = 1.58 \text{ fm}^{-1} \]

\[ \langle E'i'JT | \tilde{H}_\alpha - T_{\text{int}} | EiJ \rangle \]
\[ J^\pi = \frac{1}{2}^+, T = \frac{1}{2}, \hbar \Omega = 24 \text{ MeV} \]

NCSM ground state $^3\text{H}$

pre-diagonalization of Hamiltonian

acceleration of convergence in many-body calculations

[Graph showing pre-diagonalization and NCSM ground state]
SRG Evolution in A-Body Space

- SRG induces **irreducible** many-body contributions
  \[
  U_\alpha^\dagger H U_\alpha = \tilde{\mathcal{H}}^{[2]}_\alpha + \tilde{\mathcal{H}}^{[3]}_\alpha + \cdots + \tilde{\mathcal{H}}^{[A]}_\alpha
  \]

- restricted to a SRG evolution in 2B or 3B space

- formal **violation of unitarity**

**SRG-evolved Hamiltonians**

- **NN only**: start with NN initial Hamiltonian and evolve in two-body space
- **NN+3N-induced**: start with NN initial Hamiltonian and evolve in three-body space
- **NN+3N-full**: start with NN+3N initial Hamiltonian and evolve in three-body space

\(\alpha\)-variation provides a **diagnostic tool** to assess the contributions of omitted many-body interactions
From Jacobi to $JT$-Coupled Scheme

Transformed interaction in 3B-Jacobi basis

First problem
Many-body calculations ($A > 6$) in Jacobi coordinates not feasible → advantageous to use $m$-scheme

Second problem
$m$-scheme matrix elements become intractable for $N_{\text{max}} > 8$ (p-shell)

transformation from Jacobi into $JT$-coupled scheme

Key to efficient NCSM calculations up to $N_{\text{max}} = 14$ for p-shell nuclei

decoupling on the fly

Ab-initio many-body calculation
**$\mathcal{J}T$-Coupled Scheme vs. $m$-Scheme**

- **$m$-scheme**

  $$| (n_a l_a, s_a) j_a m_a, (n_b l_b, s_b) j_b m_b, (n_c l_c, s_c) j_c m_c; t_a m_{ta}, t_b m_{tb}, t_c m_{tc} \rangle$$

- **$\mathcal{J}T$-coupled scheme**

  $$| \{ [(n_a l_a, s_a) j_a, (n_b l_b, s_b) j_b] j_{ab}, (n_c l_c, s_c) j_c \} \mathcal{JM}; [(t_a, t_b) t_{ab}, t_c] T_{MT} \rangle$$

- **explicit consideration of interaction properties in $\mathcal{J}T$-coupled scheme**

  - Hamiltonian connects only equal $\mathcal{J}$ and $T$
  
  - **memory needs decreases** by two orders of magnitude
No-Core Shell Model (NCSM)

- **solve eigenvalue problem**: \( H |\psi_n\rangle = E_n |\psi_n\rangle \)

- **many-body basis**: Slater determinants \( |\Phi_\nu\rangle \) composed of harmonic oscillator single-particle states

\[
|\psi_n\rangle = \sum_\nu C^n_\nu |\Phi_\nu\rangle
\]

- **model space**: spanned by \( m \)-scheme states \( |\Phi_\nu\rangle \) with unperturbed excitation energy of up to \( N_{\text{max}} \hbar \Omega \)

**problem of NCSM**

enormous increase of model space with particle number \( A \) and \( N_{\text{max}} \)
Importance-Truncated NCSM

■ start with **reference state** $|\Psi_{\text{ref}}\rangle$ as approximation of target state $|\Psi_n\rangle$ from limited reference space $\mathcal{M}_{\text{ref}}$

■ a priori determination of relevant basis states $|\phi_\nu\rangle \notin \mathcal{M}_{\text{ref}}$ via first-order perturbation theory

\[
\kappa_\nu = -\frac{\langle \Phi_\nu | H_{\text{int}} | \Psi_{\text{ref}} \rangle}{\varepsilon_\nu - \varepsilon_{\text{ref}}}
\]

■ **importance truncated space** $\mathcal{M}(\kappa_{\text{min}})$ spanned by basis states with $|\kappa_\nu| \geq \kappa_{\text{min}}$

■ **solving eigenvalue problem** in $\mathcal{M}(\kappa_{\text{min}})$ provides improved approximation for target state

■ **extrapolation** of $\kappa_{\text{min}} \rightarrow 0$ considers effect of omitted contributions

■ provides **same results** as the full NCSM keeping all its advantages

■ expands **application range** to higher $A$
$^4\text{He}: \text{Ground-State Energies}$

**NN only**
- **strong $\alpha$-dependence:** induced 3N interactions
- $\hbar \Omega = 20 \text{ MeV}$

**NN+3N-induced**
- **no $\alpha$-dependence:** no induced 4N interactions

**NN+3N-full**
- **no $\alpha$-dependence:** no induced 4N interactions

- $\alpha = 0.04 \text{ fm}^4$
  - $\lambda = 2.24 \text{ fm}^{-1}$
- $\alpha = 0.05 \text{ fm}^4$
  - $\lambda = 2.11 \text{ fm}^{-1}$
- $\alpha = 0.0625 \text{ fm}^4$
  - $\lambda = 2.00 \text{ fm}^{-1}$
- $\alpha = 0.08 \text{ fm}^4$
  - $\lambda = 1.88 \text{ fm}^{-1}$
- $\alpha = 0.16 \text{ fm}^4$
  - $\lambda = 1.58 \text{ fm}^{-1}$
$^6\text{Li}: \text{Ground-State Energies}$

$E \left[ \text{MeV} \right]$

$N_{\text{max}}$

$\tilde{\hbar}\Omega = 20 \text{ MeV}$

$\alpha = 0.04 \text{ fm}^4$
$\lambda = 2.24 \text{ fm}^{-1}$

$\alpha = 0.05 \text{ fm}^4$
$\lambda = 2.11 \text{ fm}^{-1}$

$\alpha = 0.0625 \text{ fm}^4$
$\lambda = 2.00 \text{ fm}^{-1}$

$\alpha = 0.08 \text{ fm}^4$
$\lambda = 1.88 \text{ fm}^{-1}$

$\alpha = 0.16 \text{ fm}^4$
$\lambda = 1.58 \text{ fm}^{-1}$
**12C: Ground-State Energies**

### NN only

- No α-dependence: no induced 4N contrib.

### NN+3N-induced

- Some α-dependence: induced 4N interactions

### NN+3N-full

- E[MeV]

<table>
<thead>
<tr>
<th>N_max</th>
<th>E[MeV]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-60</td>
</tr>
<tr>
<td>4</td>
<td>-70</td>
</tr>
<tr>
<td>6</td>
<td>-80</td>
</tr>
<tr>
<td>8</td>
<td>-90</td>
</tr>
<tr>
<td>10</td>
<td>-100</td>
</tr>
<tr>
<td>12</td>
<td>-110</td>
</tr>
<tr>
<td>∞</td>
<td></td>
</tr>
</tbody>
</table>

**Parameters:**

- \(\alpha = 0.04 \text{ fm}^4\)
- \(\lambda = 2.24 \text{ fm}^{-1}\)

- \(\alpha = 0.05 \text{ fm}^4\)
- \(\lambda = 2.11 \text{ fm}^{-1}\)

- \(\alpha = 0.0625 \text{ fm}^4\)
- \(\lambda = 2.00 \text{ fm}^{-1}\)

- \(\alpha = 0.08 \text{ fm}^4\)
- \(\lambda = 1.88 \text{ fm}^{-1}\)

- \(\alpha = 0.16 \text{ fm}^4\)
- \(\lambda = 1.58 \text{ fm}^{-1}\)
$^{16}$O: Ground-State Energies

**NN only**

- No $\alpha$-dependence: no induced 4N contrib.

**NN+3N-induced**

- Sizable $\alpha$-dependence: induced 4N interactions

**NN+3N-full**

- $\hbar\Omega = 20$ MeV

**Parameters**

- $\alpha = 0.04$ fm$^4$
  - $\lambda = 2.24$ fm$^{-1}$
- $\alpha = 0.05$ fm$^4$
  - $\lambda = 2.11$ fm$^{-1}$
- $\alpha = 0.0625$ fm$^4$
  - $\lambda = 2.00$ fm$^{-1}$
- $\alpha = 0.08$ fm$^4$
  - $\lambda = 1.88$ fm$^{-1}$
- $\alpha = 0.16$ fm$^4$
  - $\lambda = 1.58$ fm$^{-1}$
Spectroscopy of $^{12}\text{C}$

- **NN only**
- **NN+3N-induced**
- **NN+3N-full**

$\hbar \Omega = 16 \text{ MeV}$

$$\alpha = 0.04 \text{ fm}^4 \quad \lambda = 2.24 \text{ fm}^{-1}$$

$$\alpha = 0.08 \text{ fm}^4 \quad \lambda = 1.88 \text{ fm}^{-1}$$

spectra largely insensitive to induced $4N$
SRG Model Space & Frequency Conversion

Roth, AC, Langhammer et al. — in preparation
accelerate convergence by \textbf{pre-diagonalizing} the Hamiltonian with respect to the many-body basis

- \textbf{unitary} transformation driven by

\[
\frac{d}{d\alpha} \langle E'i'JT \mid \tilde{H}_\alpha \mid EiT \rangle \approx (2\mu)^2 \sum_{E'',E'''} \sum_{i'',i'''} \langle E'i'JT \mid T_{\text{int}} \mid E''i''JT \rangle \langle E''i''JT \mid \tilde{H}_\alpha \mid E'''i'''JT \rangle \langle E'''i'''JT \mid \tilde{H}_\alpha \mid EiT \rangle \\
- 2 \langle E'i'JT \mid \tilde{H}_\alpha \mid E''i''JT \rangle \langle E''i''JT \mid T_{\text{int}} \mid E'''i'''JT \rangle \langle E'''i'''JT \mid \tilde{H}_\alpha \mid EiT \rangle \\
+ \langle E'i'JT \mid \tilde{H}_\alpha \mid E''i''JT \rangle \langle E''i''JT \mid \tilde{H}_\alpha \mid E'''i'''JT \rangle \langle E'''i'''JT \mid T_{\text{int}} \mid EiT \rangle
\]

SRG model space truncated \( E \leq E_{\text{(SRG) max}} \)
SRG Model Space

- large angular momenta less important for low-energy properties

- $J$-dependent model space truncation $E_{\text{max}}^{(\text{SRG})}(J)$

**SRG model-space ramp**

- use $A$-ramp as standard
- use $B$- and $C$-ramp to investigate sensitivity to model space truncation
**Frequency Conversion: $^{16}$O Ground State**

**standard SRG**

- Physical content of SRG model space depends on $\hbar \Omega$
- SRG model space insufficient for low $\hbar \Omega$
  - especially for increasing mass number

**Idea:**

- SRG transformation for adequate $\hbar \tilde{\Omega}$
- Convert to $\hbar \Omega$ needed for the **many-body calculations**
Frequency Conversion: $^{16}$O Ground State

**standard SRG**

$E [\text{MeV}]$

-80
-90
-100
-110
-120
-130
-140
-150

$\Delta E [\text{MeV}]$

8
6
4
2
0

$\hbar \Omega [\text{MeV}]$

12 14 16 18 20 22 24

**Frequency Conversion**

$E [\text{MeV}]$

-80
-90
-100
-110
-120
-130
-140
-150

$\Delta E [\text{MeV}]$

8
6
4
2
0

$\hbar \bar{\Omega} = 20 \text{ MeV}$

NN+3N-full

$\alpha = 0.08 \text{ fm}^4$

NN+3N-full

$\alpha = 0.08 \text{ fm}^4$

$\hbar \bar{\Omega}$ = 20 MeV
Frequency Conversion: $^{16}\text{O}$ Ground State

**standard SRG**

$\text{NN+3N-full}$

$\alpha = 0.08 \text{ fm}^4$

$E [\text{MeV}]$ vs $\hbar \Omega [\text{MeV}]$

- $A$-ramp
- $B$-ramp
- $C$-ramp

$\Delta E [\text{MeV}]$ vs $\hbar \Omega [\text{MeV}]$

$\hbar \tilde{\Omega} = 24 \text{ MeV}$

Insensitive to model space truncation
Sensitivity of Nuclear Spectra on Chiral 3N Interactions

Roth, Langhammer, AC et al. — in preparation
analyze the sensitivity of spectra on **low-energy constants** \((c_i, c_D, c_E)\) and **cutoff** \((\Lambda)\) of the chiral 3N interaction at \(N^2\)LO

why this is interesting:

- **impact of \(N^3\)LO contributions**: some \(N^3\)LO diagrams can be absorbed into the \(N^2\)LO structure by shifting the \(c_i\) constants

\[
\tilde{c}_1 = c_1 - \frac{g_A^2 M_\pi}{64\pi F_\pi^2}, \quad \tilde{c}_3 = c_3 + \frac{g_A^4 M_\pi}{16\pi F_\pi^2}, \quad \tilde{c}_4 = c_4 - \frac{g_A^4 M_\pi}{16\pi F_\pi^2}
\]

(Bernard et al., Ishikawa, Robilotta)

- **uncertainty propagation**: sizable variations of the \(c_i\) from different extractions (also affects NN)

\[
c_1 = -1.23 ... -0.76, \quad c_3 = -5.91 ... -4.05, \quad c_4 = 3.04 ... 5.40\quad [\text{GeV}^{-1}]
\]

- **cutoff dependence**: does the cutoff choice in the 3N interaction affect nuclear structure observables?
analyze the sensitivity of spectra on **low-energy constants** (\(c_i, c_D, c_E\)) and **cutoff** (\(\Lambda\)) of the chiral 3N interaction at N\(^2\)LO

<table>
<thead>
<tr>
<th></th>
<th>(C_1) [GeV(^{-1})]</th>
<th>(C_3) [GeV(^{-1})]</th>
<th>(C_4) [GeV(^{-1})]</th>
<th>(C_D)</th>
<th>(C_E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard 3N</td>
<td>-0.81</td>
<td>-3.2</td>
<td>+5.4</td>
<td>-0.2</td>
<td>-0.205</td>
</tr>
<tr>
<td>(c_i) shifted</td>
<td>-0.94</td>
<td>-2.3</td>
<td>+4.5</td>
<td>-0.2</td>
<td>-0.085</td>
</tr>
<tr>
<td>(c_1) shifted</td>
<td>-0.94</td>
<td>-3.2</td>
<td>+5.4</td>
<td>-0.2</td>
<td>-0.247</td>
</tr>
<tr>
<td>(c_3) shifted</td>
<td>-0.81</td>
<td>-2.3</td>
<td>+5.4</td>
<td>-0.2</td>
<td>-0.200</td>
</tr>
<tr>
<td>(c_4) shifted</td>
<td>-0.81</td>
<td>-3.2</td>
<td>+4.5</td>
<td>-0.2</td>
<td>-0.130</td>
</tr>
<tr>
<td>(c_D = -1)</td>
<td>-0.81</td>
<td>-3.2</td>
<td>+5.4</td>
<td>-1.0</td>
<td>-0.386</td>
</tr>
<tr>
<td>(c_D = +1)</td>
<td>-0.81</td>
<td>-3.2</td>
<td>+5.4</td>
<td>+1.0</td>
<td>-0.038</td>
</tr>
<tr>
<td>(\Lambda = 400) MeV</td>
<td>-0.81</td>
<td>-3.2</td>
<td>+5.4</td>
<td>-0.2</td>
<td>+0.098</td>
</tr>
<tr>
<td>(\Lambda = 450) MeV</td>
<td>-0.81</td>
<td>-3.2</td>
<td>+5.4</td>
<td>-0.2</td>
<td>-0.016</td>
</tr>
</tbody>
</table>

refit \(c_E\) parameter to reproduce \(^4\)He ground-state energy
$^{12}\text{C} \, : \, \text{Sensitivity on } c_i$

- Many states are rather $c_i$-insensitive.
- First $1^+$ state shows strong $c_3$-sensitivity.

$\hbar \Omega = 16 \text{ MeV}$

$N_{\text{max}} = 8$

$\alpha = 0.08 \text{ fm}^4$
$^{12}$C : Sensitivity on $c_D$ & Cutoff

<table>
<thead>
<tr>
<th>$c_D = -1.0$</th>
<th>$c_D = -0.2$</th>
<th>$c_D = +1.0$</th>
<th>$\Lambda = 400$ MeV</th>
<th>$\Lambda = 450$ MeV</th>
<th>$\Lambda = 500$ MeV</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>no 3N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- weak dependence on $c_D$, stronger dependence on $\Lambda$
- again first $1^+$ state is most sensitive

$\hbar \Omega = 16$ MeV
$N_{\text{max}} = 8$
$\alpha = 0.08$ fm$^4$
Correlation Analysis

$^{12}\text{C}(1^+)$ vs. $^{10}\text{B}(1^+)$

$^{10}\text{B}: E(1^+) - E(0^+) [\text{MeV}]$

$\hbar\Omega = 16 \text{ MeV}$

$N_{\text{max}} = 8$

$\alpha = 0.08 \text{ fm}^4$

$^{11}\text{B}(1/2^-)$ vs. $^9\text{Be}(1/2^-)$

$^9\text{Be}: E(1/2^-) - E(3/2^-) [\text{MeV}]$

not compatible with experiment:
new operator structures necessary?

$N^3\text{LO, N}^4\text{LO, }\Delta\text{-full …}$
Towards Next-Generation Chiral Hamiltonians
Technical Aspects

- **starting point**: numerical 3N matrix elements in partial-wave Jacobi-momentum basis (antisym. under $1 \leftrightarrow 2$)

$$\langle p_1' p_2' \beta' | V_3 (1 + P) | p_1 p_2 \beta \rangle \quad \text{or} \quad \langle p_1' p_2' \beta' | (1 + P) V_3 (1 + P) | p_1 p_2 \beta \rangle$$

$$| p_1 p_2 \beta \rangle = | p_1 p_2 \{ (L_1, S_1)_j; (L_2, S_2)_{j'} \} J M_j; (T_1, T_2) T M_T \rangle$$

- numerical partial-wave decomposition of Skibinski et al.
- ongoing collaborative effort to produce $N^2$LO/$N^3$LO matrix elements (Cracow, Bochum, Bonn, Ohio SU, Iowa SU, Darmstadt)

- **need** transformation to **HO basis** for nuclear structure calculations!
  - SRG in momentum space then transformation to HO basis (Kai Hebeler)
  - direct transformation to HO basis
Machinery 3-Body Momentum Basis

**Our Strategy:**

- transform initial interaction to antisym. HO Jacobi basis
- use HO machinery afterwards (SRG; $J$, $T$-coupled scheme; ...)
  - SRG in HO basis very efficient (discrete, consider antisymmetry)
  - new developments in HO basis applicable for all chiral interactions

**first application:** consistent NN+3N Hamiltonian at $N^2$LO

- NN at $N^2$LO: Epelbaum et al., cutoffs 450,...,600 MeV, phase-shift fit $\chi^2$/dat $\sim$ 10 ($\sim$ 1) up to 300 MeV (100 MeV)
- 3N at $N^2$LO: Epelbaum et al., cutoffs 450,...,600 MeV, nonlocal, fit to $a(nd)$ and $E(^3H)$, included up to $J=7/2$
\[ \hbar \Omega = 16 \text{ MeV} \]
\[ N_{\text{max}} = 8 \]
\[ \alpha = 0.08 \text{ fm}^4 \]
$^{12}$C: Consistent $N^2$LO Hamiltonians

![Graph showing energy levels for different Hamiltonians and a consistent description with different NN+3N Hamiltonians.]

- Standard N3LO+N2LO 500 MeV
- Epelbaum N2LO+N2LO 450/500 MeV
- Epelbaum N2LO+N2LO 550/600 MeV

$\hbar \Omega = 16$ MeV

$N_{\text{max}} = 8$

$\alpha = 0.08 \text{ fm}^4$
$^{10}\text{B : Consistent } N^2\text{LO Hamiltonians}$

- **standard**
  - N3LO+N2LO
  - 500 MeV

- **Epelbaum**
  - N2LO+N2LO
  - 450/500 MeV
  - 550/600 MeV

- **Epelbaum**
  - N2LO+N2LO
  - 450/500 MeV
  - 550/600 MeV

- Large variations at NN level
- More consistent description with NN+3N

\[ \hbar \Omega = 16 \text{ MeV} \]
\[ N_{\text{max}} = 8 \]
\[ \alpha = 0.08 \text{ fm}^4 \]
$^{10}$B: Consistent $N^2$LO Hamiltonians

- Large variations at NN level
- More consistent description with NN+3N

$$\hbar \Omega = 16 \text{ MeV}$$
$$N_{\text{max}} = 8$$
$$\alpha = 0.08 \text{ fm}^4$$
Correlation Analysis: $^{12}\text{C}(1^+) \text{ vs. } ^{10}\text{B}(1^+)$

\[ C(1^+) = (1 + \frac{1}{2}) \text{ vs. } B(1^+) = (1 + \frac{1}{2}) \]

\[ B = E(1^+) - E(3^+) \text{ [MeV]} \]

\[ C = E(1^+) - E(0^+) \text{ [MeV]} \]

- $\hbar\Omega = 16$ MeV
- $N_{\text{max}} = 8$
- $\alpha = 0.08$ fm$^4$

+ exp

Entem&Machleidt+Navrátil
- no 3N
- std 3N
- $c_i$ var
- $c_D$ var
- $\Lambda$ var
Correlation Analysis: $^{12}\text{C}(1^+) \text{ vs. } ^{10}\text{B}(1^+)$

$$\hbar\Omega = 16 \text{ MeV}$$
$$N_{\text{max}} = 8$$
$$\alpha = 0.08 \text{ fm}^4$$

- Interesting deviation from E&M+N systematics
- NN possible reason

Entem&Machleidt+Navrátil
- no 3N
- std 3N
- $c_i$ var
- $c_D$ var
- $\Lambda$ var

Epelbaum @ N$^2$LO
- NN+3N

$^{12}\text{C} : E(1^+) - E(0^+) \text{ [MeV]}$
$^{10}\text{B} : E(1^+) - E(3^+) \text{ [MeV]}$
SRG in Four-Body Space
Induced Four-Body Contributions

- **is there a direct solution?**
  - alternative generator for the SRG
    - so far found only trade-offs between induced 4N & convergence acceleration
  - SRG in four-body space

**Graphs:**

- **16O**
- **NN+3N**
- **$\hbar\Omega = 20$ MeV**

**Energy Levels (E [MeV]):**

- **500 MeV**
  - $c_D = -0.2$
  - $c_E = -0.205$

- **450 MeV**
  - $c_D = -0.2$
  - $c_E = -0.016$

- **400 MeV**
  - $c_D = -0.2$
  - $c_E = 0.098$

- **350 MeV**
  - $c_D = -0.2$
  - $c_E = 0.205$
Four-Body Jacobi Basis

- Jacobi coordinate: \[ \xi_3 = \sqrt{\frac{3}{4}} \left( \frac{1}{2} (\vec{r}_a + \vec{r}_b + \vec{r}_c) - \vec{r}_d \right) \]

- Jacobi state antisym. under \(1 \leftrightarrow 2 \leftrightarrow 3\) (extension of antisym. three-body Jacobi state)

\[ |E_{12}E_3 i_{12}; \alpha \rangle = |E_{12}E_3 i_{12} [J_{12}, (L_3, S_3)J_3]JM_j; (T_{12}T_3)TM_T \rangle \]

- Antisym. Jacobi state

\[ |E iJM_J TM_T \rangle = \sum_{i_{12}, \beta} \tilde{c}_{\alpha,i}^{E_{12},E_3,i_{12}} |E_{12}E_3 i_{12}; \alpha \rangle \quad \text{with} \quad E = E_{12} + E_3 \]

introduce **four-body CFPs**: \( \tilde{c}_{E_{12},E_3,i_{12}}^{\alpha,i} \)
SRG Evolution in Four-Body Space

4B-Jacobi HO matrix elements

\[ \alpha = 0.16 \text{ fm}^4 \]

\[ \lambda = 1.58 \text{ fm}^{-1} \]

\[ \langle E'i'JT | \tilde{H}_\alpha - T_{\text{int}} | EiT \rangle \]

\[ J^\pi = 0^+, T = 0, \hbar \Omega = 24 \text{ MeV} \]

NCSM ground state \(^4\text{He}\)

moderate pre-diagonalization of Hamiltonian

acceleration of convergence in many-body calculations

\( N_{\text{max}} \)
First Shot: Sum over Fourth Particle

- transformation to **four-body m-scheme** basis and additional **normal-ordering** approximation in progress

- meanwhile: create **effective three-body interaction** in Jacobi basis

  - sum over fourth particle (unperturbed m-scheme state)
  - only consider equal $J_{12}, T_{12}$ in Bra and Ket and average over projections
  - set three-body center of mass motion to ground-state

\[
\langle E'_{12} i'_{12} J_{12} T_{12} | \hat{V}_{3N}^{\text{eff}} | E_{12} i_{12} J_{12} T_{12} \rangle \\
= \frac{1}{4} \sum_M \sum_{n_d \lambda_d \bar{d}} \langle 000 | \otimes | E_{12} i_{12} J_{12} M J_{12} M T_{12} M | \hat{V}_{4N} \times \{ |000 \rangle \otimes | E_{12} i_{12} J_{12} / \rangle \rangle \rangle |

\textbf{Motivation:} reproduces ground-state energy for closed shell nuclei in $N_{\text{max}} = 0$ space
First Shot: $^{16}\text{O}$ Ground State

$\hbar\Omega = 24 \text{ MeV}$

$N_{\text{max}}$ vs $E$ [MeV]

$\alpha = 0.04 \text{ fm}^4$  
$\lambda = 2.24 \text{ fm}^{-1}$  
$\alpha = 0.08 \text{ fm}^4$  
$\lambda = 1.88 \text{ fm}^{-1}$

NN+3N-std
First Shot: $^{16}$O Ground State

\[ \hbar \Omega = 24 \text{ MeV} \]

- correction by induced 4N in right direction, but too small
- improvements:
  - consider further 4N channels
  - increase $E^{(SRG)}_{\text{max}}$
  - use normal-ordering approximation

<table>
<thead>
<tr>
<th>$N_{\text{max}}$</th>
<th>$E$ [MeV]</th>
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<tbody>
<tr>
<td>2</td>
<td>-130.0</td>
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<tr>
<td>3</td>
<td>-129.5</td>
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<td>-126.5</td>
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<tr>
<td>10</td>
<td>-126.0</td>
</tr>
</tbody>
</table>

- $\alpha = 0.04 \text{ fm}^4$
- $\alpha = 0.08 \text{ fm}^4$
- $\lambda = 2.24 \text{ fm}^{-1}$
- $\lambda = 1.88 \text{ fm}^{-1}$
Conclusions
Conclusions

- **SRG** evolution in **HO basis** efficient and **improvable**
  - frequency conversion & model space increase

- **consistent four-body** SRG evolution
  (for induced and initial contributions)
  - inclusion via **effective three-body** interaction
  - next step: use normal-ordering approximation

- **p-shell spectra** provide powerful testbed for chiral potentials

- machinery ready to use **3N @ N^3LO** in momentum Jacobi basis
  - directly applicable in IT-NCSM, CC, IM-SRG, RGM ...
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**LENPIC**

Low-Energy Nuclear Physics International Collaboration