Interplay of Collective and Single Particle Modes in the Pairing Properties of Open Shell Nuclei

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INT workshop 8-12 April 2013 - , Seattle
A. Overview of Mean Field Pairing calculations (HFB)
   Including v-14/18 as pairing force
B. PVC and Single-particle self-energy
C. Phonon Exchange Induced Pairing Interaction (Old theory)
D. PVC and Quasi-particle self-energy or SCGF (New theory)
   Contact with Ab-Initio approaches to open-shell nuclei
Typical forces employed in finite nuclei pairing HFB-like calculations:

I) Monopole: \[ H_p = -G \ P^+P \] with \( G = -25/A \text{ MeV} \)
being \( P^+ = \sum_{m>0} a^+_m a^+_{-m} \); restricted to one shell

ii) Contact interactions \( V_c(1-\eta^*\rho(r) / \rho_0) \) \( (E_{\text{cut}}=60\text{MeV}) \)

Commonly used with Skyrme forces

iii) Gogny force
Continuum is relevant

iv) Argonne int.
Continuum is crucial
Up to 800MeV

v) Nuclear Matter Calculation Based Pairing Interaction
Show some Illustrative results
Contact interaction


\[
V(r-r') = V_c \left(1 - \eta^* \frac{\rho(r)}{\rho_0}\right) \delta^3(r-r')
\]

\(\eta = 0; 0.5 \text{ and } 1.0 (\text{vol.; mix.; surf.})\)

\((E_{\text{cut}} = 60\text{MeV});\)

**TABLE IV:** RMS residuals of \(\Delta_0^{(3)}\) obtained in various models. All energies are in MeV. The last column shows the ratio of proton and neutron effective pairing strengths obtained through the optimization procedure. The mass predictions of the HFB-14 model [16] were taken from [51].

<table>
<thead>
<tr>
<th>Theory</th>
<th>pairing</th>
<th>residual neutrons</th>
<th>residual protons</th>
<th>(V_0^{\text{eff}}(p)/V_0^{\text{eff}}(n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td></td>
<td>0.31</td>
<td>0.27</td>
<td></td>
</tr>
<tr>
<td>(c/A^{\alpha})</td>
<td></td>
<td>0.24</td>
<td>0.22</td>
<td></td>
</tr>
<tr>
<td>HF + BCS</td>
<td>volume</td>
<td>0.31</td>
<td>0.38</td>
<td>1.05</td>
</tr>
<tr>
<td>HF + BCS</td>
<td>mixed</td>
<td>0.30</td>
<td>0.36</td>
<td>1.08</td>
</tr>
<tr>
<td>HF + BCS</td>
<td>surface</td>
<td>0.27</td>
<td>0.35</td>
<td>1.12</td>
</tr>
<tr>
<td>HFB</td>
<td>mixed</td>
<td>0.27</td>
<td>0.32</td>
<td>1.11</td>
</tr>
<tr>
<td>HFB + LN</td>
<td>mixed</td>
<td>0.23</td>
<td>0.28</td>
<td>1.11</td>
</tr>
<tr>
<td>HFB-14</td>
<td></td>
<td>0.46</td>
<td>0.44</td>
<td>1.10</td>
</tr>
</tbody>
</table>
Pairing gaps in the Hartree-Fock-Bogoliubov theory with the Gogny D1S interaction

**FIG. 1.** (Color online) Neutron pairing gaps $\Delta^{(3)}$ in the oxygen isotope chain. Energies were computed in the $N_{sh} = 10$ harmonic oscillator space.

**FIG. 2.** (Color online) Neutron pairing gaps $\Delta^{(3)}$ in the Sn isotope chain. Energies were computed in the $N_{sh} = 12$ harmonic oscillator space.
Another correlation is associated with the polarization of the nucleus by the valence nucleons. The resulting induced pairing interaction has been calculated to give as strong a contribution as the nucleon-nucleon interaction in the pairing channel \cite{32}, Such induced interactions are long-ranged and energy-dependent, and vary from nucleus to nucleus depending on its structure. It would not be surprising that the effects were beyond the scope of the simple energy functionals in current use.
Argonne (1S₀)
Barranco, Broglia, Esbensen, Vigezzi, PLB(1997)

Neutron saturated nucleus (Z=50) immersed in a superfluid neutron sea

We have repeated the calculations, but this time setting the Fermi energy at −7.2 MeV, that is, for the isolated atomic nucleus $^{120}$Sn. The value of $R_{\text{box}}$ has been varied within the range 8–15 fm to check the stability of the results. The values of the pairing gaps become stable once $R_{\text{box}} \geq 12$ fm. The results of the calculation with $R_{\text{box}} = 15$ fm are displayed in Fig. 1(b). The value of the pairing gap at the Fermi energy is $2.2^{+0.4}_{-0.8}$ MeV, the “errors” reflecting the conspicuous state dependence of $\Delta$. This value is about 50% higher than the experimental value of 1.47 MeV deduced from the odd-even mass differences [22]. This also in keeping with the fact that the free nucleon-nucleon interaction seems to provide too strong pairing correlations in finite nuclei (cf. e.g. [23]). In any case, theory provides a sound framework for an eventual quantitative description of pairing in nuclei.

$m=1.0 \ m_0 \Rightarrow \text{Too large gap}$
Finite size effects on the pairing field (BCS with the bare force)

Potential in the Wigner cell

Pairing gap in uniform neutron matter

\( \varepsilon_F = 13.5 \text{MeV} \)

P.M. Pizzochero, F. Barranco, E. Vigezzi, R.A. Broglia, APJ 569(2003)381
Spatial dependence of pairing densities and pairing gaps

FINITE NUCLEI, FINITE RANGE FORCE

HFB Equations are expanded on a basis

\[
\begin{pmatrix}
    h_{nn'lj} - \lambda & \Delta_{nn'lj} \\
    \Delta_{nn'lj} & -h_{nn'lj} + \lambda
\end{pmatrix}
\begin{pmatrix}
    U_{nlj}^\alpha \\
    V_{nlj}^\alpha
\end{pmatrix}
= E_{lj}^\alpha
\begin{pmatrix}
    U_{nlj}^\alpha \\
    V_{nlj}^\alpha
\end{pmatrix}
\]

\[\kappa(R_{cm}, r_{12}) = \frac{1}{8\pi} \sum_{nn'lj\alpha} (2j + 1) U_{nlj}^\alpha V_{nlj}^\alpha \varphi_{nlj}^* (r^{'1}_{12}) \varphi_{n'lj}(r^{'2}_{12}) P_l (\cos(\vartheta_{12}))\]

\[\Delta(R_{cm}, r_{12}) = -V_{\text{eff}}(r^{'1}_{12}) \kappa(R_{cm}, r_{12})\]

\[\Delta(R_{cm}, k) = \int dr_{12} \Delta(R_{cm}, r_{12}) \exp(ikr^{'1}_{12})\]
Spatial description of (non-local) pairing gap
Essential for a consistent description of vortex pinning!

The range of the force is small compared to the coherence length, but not compared to the diffusivity of the nuclear potential.

The local-density approximation overestimates the decrease of the pairing gap in the interior of the nucleus. (PROXIMITY EFFECTS)
V_low-k pairing gaps in Sly4 HF field

Dependence on the effective mass at high momenta? (V_low-k vs V_18)

Mean field calculation with V_low-k pairing force: 3-body force reduces the pairing gaps


120Sn with V_18 and Gogny HF-field: 1.1MeV
Results depend on the Central Potential parameters, including nucleon effective mass.

Any way some solid conclusions can be extracted

1. Bare Argonne is tractable at hfB level

2. With “reasonable” potentials as Sly4 a close to experiment pairing gap is obtained: 1.1MeV vs 1.3MeV.

3. Using V_low-K (and NNN forces) similar results are obtained

Resonable potential are related to a $m^*=0.7m_o$

which also leave room to self-energy corrections ($m_{exp}=1.0m_0$)
Beyond mean field needed?

a). Neutron matter shows important screening

b). Single-particles are importantly affected, why not quasi-particle properties like the pairing gap?

c). In B&M-II book an estimate was made...

d). If simple DF are used: 

(Robledo, Bernard, Bertsch, PRC86(2012))

Another correlation is associated with the polarization of the nucleus by the valence nucleons. The resulting induced pairing interaction has been calculated to give as strong a contribution as the nucleon-nucleon interaction in the pairing channel [32], Such induced interactions are long-ranged and energy-dependent, and vary from nucleus to nucleus depending on its structure. It would not be surprising that the effects were beyond the scope of the simple energy functionals in current use. (*)

(*)Alternative sophisticated DF: I. Stetcu (LANL, 2013 talk)
In second order, the particle-vibration coupling gives rise to an interaction between two particles, which can be evaluated in a manner similar to the particle-phonon interaction considered in Sec. 6-5d. To illustrate the magnitude of the polarization force, we consider the limiting case in which the frequency of the exchanged phonon is large compared to the energy differences between the particle states. In this case, one can view the interaction as resulting from the static deformation (6-217) produced by the first particle acting on the second. Thus, for a mode of multipolarity $\lambda$, one obtains (see Eq. (6-68))

$$V_\lambda(1,2) = -\frac{2\lambda+1}{4\pi C_\lambda} k_\lambda(r_1) k_\lambda(r_2) P_\lambda(\cos \vartheta_{12})$$

(6-228)

The order of magnitude of the polarization interaction (6-228) is given by $f^2 \hbar \omega_\lambda$ (as for the particle self-energy). For the high-frequency modes, we have $f^2 \hbar \omega_\lambda \sim \varepsilon_{PA}^{-1}$, which is comparable to the average interaction between nucleons in the nucleus ($\sim V_0 A^{-1}$). The magnitude of the polarization interaction can be seen directly from the fact that the deformation of the closed shells produced by a single particle implies polarization moments comparable with the bare moments of the particles (see p. 510); hence, the corresponding contribution to the polarization interaction (6-228) is similar in magnitude to the direct force.

The polarization interaction resulting from the coupling to the low-frequency modes may be considerably larger than the bare force; since the frequencies of these modes may be comparable with the particle frequencies, it may be necessary to go beyond the static approximation (6-228), as in the evaluation of the particle-phonon interaction.
B. Particle Vibration Coupling

and

Single-particle Self-energy
Effective charges

One-particle transfer leading to Vibrational states

The vertex: Basic Ingredient

Simple Collective Model value

\[ h_{j,j',\lambda} = \beta \lambda (2\lambda+1)^{-1/2} <j||R_o \frac{dU}{dr} Y_\lambda||j'> (2j+1)^{-1/2} \]
The paradigmatic case; $^{208}\text{Pb}(3^-) + 1\text{p}(h_{9/2})$

![Diagram of shell-model and vibrational states with energy levels and transitions](image)

**Fig. 11.** Observed low-lying energy levels of $^{209}\text{Bi}$. The coupling of different excitation modes are discussed in the text.

**Table 6**

Energy shifts of the septuplet members ($h_{9/2} 3^- f^{\pi}$) in $^{209}\text{Bi}$ from the unperturbed vibrational frequency 2.614 MeV. Experimental values are taken from ref. [29]

<table>
<thead>
<tr>
<th>$I$</th>
<th>$\delta E_{\text{exp}}$ (keV)</th>
<th>$\delta E_{\text{calc}}$ (keV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/2</td>
<td>-120</td>
<td>+36 $\rightarrow$ -190</td>
</tr>
<tr>
<td>5/2</td>
<td>+4</td>
<td>+7</td>
</tr>
<tr>
<td>7/2</td>
<td>-29</td>
<td>-6</td>
</tr>
<tr>
<td>9/2</td>
<td>-49</td>
<td>-89</td>
</tr>
<tr>
<td>11/2</td>
<td>-14</td>
<td>-31</td>
</tr>
<tr>
<td>13/2</td>
<td>-14</td>
<td>-63</td>
</tr>
<tr>
<td>15/2</td>
<td>+130</td>
<td>+156</td>
</tr>
</tbody>
</table>

I. Hamamoto, 1973
SELF ENERGY RENORMALIZATION OF SINGLE-PARTICLE STATES: CLOSED SHELL

\[ ^{208}\text{Pb} \]

Relativistic Mean Field Calculations
Litvinova et al, PRC84, 014305 (2011)
Full Skyrme (Sly5), including momentum dependent terms
 Colo' et al, PRC82, 064307

The vertex is deduced from the Transition Densities obtained in RPA calculation.
Similar difficulties in:

Propagation of uncertainties in the Skyrme energy-density-functional model

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2Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, ul. Hoża 69, PL-00-681 Warsaw, Poland
3Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria

(Dated: January 28, 2013)

TABLE I: The $^{208}$Pb proton (top) and neutron (bottom) s.p. energies $e_{HF}$ (b) and their standard errors (c), as compared to the empirical values (d) [25], residuals, $\Delta(e_{HF}) = e_{HF} - e_{exp}$ (e), PVCs $\delta_{PVC}$ (f), and residuals of the PVC-corrected s.p. energies $\Delta(e_{PVC}) = e_{HF} + \delta_{PVC} - e_{exp}$ (g). Where applicable, we also give the root-mean-square (rms) values of entries shown in a given column. All energies are in MeV.

<table>
<thead>
<tr>
<th>orbital</th>
<th>$e_{HF}$ (a)</th>
<th>$\sigma(e_{HF})$ (c)</th>
<th>$e_{exp}$ (d)</th>
<th>$\Delta(e_{HF})$ (e)</th>
<th>$\delta_{PVC}$ (f)</th>
<th>$\Delta(e_{PVC})$ (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi 3p_{1/2}$</td>
<td>0.219</td>
<td>0.181</td>
<td>−0.16</td>
<td>0.38</td>
<td>−0.440</td>
<td>−0.06</td>
</tr>
<tr>
<td>$\pi 3p_{3/2}$</td>
<td>−1.045</td>
<td>0.139</td>
<td>−0.68</td>
<td>−0.37</td>
<td>−0.662</td>
<td>−1.03</td>
</tr>
<tr>
<td>$2f_{5/2}$</td>
<td>−1.284</td>
<td>0.229</td>
<td>−0.97</td>
<td>−0.31</td>
<td>−0.480</td>
<td>−0.79</td>
</tr>
<tr>
<td>$\pi 1i_{13/2}$</td>
<td>−2.794</td>
<td>0.238</td>
<td>−2.10</td>
<td>−0.60</td>
<td>−0.221</td>
<td>−0.92</td>
</tr>
<tr>
<td>$\pi 1h_{9/2}$</td>
<td>−3.501</td>
<td>0.282</td>
<td>−3.80</td>
<td>0.30</td>
<td>−0.280</td>
<td>0.02</td>
</tr>
<tr>
<td>$2f_{7/2}$</td>
<td>−3.725</td>
<td>0.115</td>
<td>−2.90</td>
<td>−0.83</td>
<td>−0.284</td>
<td>−1.11</td>
</tr>
<tr>
<td>$3s_{1/2}$</td>
<td>−8.036</td>
<td>0.140</td>
<td>−8.01</td>
<td>−0.03</td>
<td>−0.108</td>
<td>−0.13</td>
</tr>
<tr>
<td>$2d_{3/2}$</td>
<td>−8.378</td>
<td>0.199</td>
<td>−8.36</td>
<td>−0.02</td>
<td>0.220</td>
<td>0.20</td>
</tr>
<tr>
<td>$1h_{11/2}$</td>
<td>−9.153</td>
<td>0.207</td>
<td>−9.36</td>
<td>0.21</td>
<td>−0.141</td>
<td>0.07</td>
</tr>
<tr>
<td>$2d_{5/2}$</td>
<td>−10.117</td>
<td>0.117</td>
<td>−9.82</td>
<td>−0.30</td>
<td>0.116</td>
<td>−0.18</td>
</tr>
<tr>
<td>$1g_{7/2}$</td>
<td>−10.908</td>
<td>0.229</td>
<td>−12.00</td>
<td>1.09</td>
<td>0.131</td>
<td>1.22</td>
</tr>
<tr>
<td>rms</td>
<td>n.a.</td>
<td>0.196</td>
<td>n.a.</td>
<td>0.52</td>
<td>0.328</td>
<td>0.70</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>orbital</th>
<th>$e_{HF}$ (a)</th>
<th>$\sigma(e_{HF})$ (c)</th>
<th>$e_{exp}$ (d)</th>
<th>$\Delta(e_{HF})$ (e)</th>
<th>$\delta_{PVC}$ (f)</th>
<th>$\Delta(e_{PVC})$ (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu 3d_{3/2}$</td>
<td>−1.856</td>
<td>0.250</td>
<td>−1.40</td>
<td>−0.46</td>
<td>−0.104</td>
<td>−0.56</td>
</tr>
<tr>
<td>$4s_{1/2}$</td>
<td>−2.051</td>
<td>0.235</td>
<td>−1.90</td>
<td>−0.15</td>
<td>−0.668</td>
<td>−0.82</td>
</tr>
<tr>
<td>$\nu 2g_{7/2}$</td>
<td>−2.141</td>
<td>0.258</td>
<td>−1.44</td>
<td>−0.70</td>
<td>−0.280</td>
<td>−0.98</td>
</tr>
<tr>
<td>$1f_{15/2}$</td>
<td>−2.231</td>
<td>0.167</td>
<td>−2.51</td>
<td>0.28</td>
<td>−0.226</td>
<td>0.05</td>
</tr>
<tr>
<td>$3d_{5/2}$</td>
<td>−2.549</td>
<td>0.166</td>
<td>−2.37</td>
<td>−0.18</td>
<td>−0.384</td>
<td>−0.56</td>
</tr>
<tr>
<td>$\nu 1i_{11/2}$</td>
<td>−2.680</td>
<td>0.223</td>
<td>−3.16</td>
<td>0.48</td>
<td>−0.271</td>
<td>0.21</td>
</tr>
<tr>
<td>$\nu 2g_{9/2}$</td>
<td>−4.336</td>
<td>0.071</td>
<td>−3.94</td>
<td>−0.40</td>
<td>−0.183</td>
<td>−0.58</td>
</tr>
<tr>
<td>$\nu 3p_{1/2}$</td>
<td>−7.855</td>
<td>0.109</td>
<td>−7.37</td>
<td>−0.49</td>
<td>0.152</td>
<td>−0.33</td>
</tr>
<tr>
<td>$\nu 3p_{3/2}$</td>
<td>−8.503</td>
<td>0.065</td>
<td>−8.26</td>
<td>−0.24</td>
<td>−0.119</td>
<td>−0.36</td>
</tr>
<tr>
<td>$2f_{5/2}$</td>
<td>−8.519</td>
<td>0.143</td>
<td>−7.94</td>
<td>−0.58</td>
<td>0.156</td>
<td>−0.42</td>
</tr>
<tr>
<td>$\nu 1i_{13/2}$</td>
<td>−8.603</td>
<td>0.162</td>
<td>−9.24</td>
<td>0.64</td>
<td>−0.130</td>
<td>0.51</td>
</tr>
<tr>
<td>$\nu 1h_{9/2}$</td>
<td>−9.922</td>
<td>0.190</td>
<td>−11.40</td>
<td>1.48</td>
<td>0.120</td>
<td>1.60</td>
</tr>
<tr>
<td>$2f_{7/2}$</td>
<td>−10.407</td>
<td>0.103</td>
<td>−9.81</td>
<td>−0.60</td>
<td>0.179</td>
<td>−0.42</td>
</tr>
<tr>
<td>rms</td>
<td>n.a.</td>
<td>0.176</td>
<td>n.a.</td>
<td>0.61</td>
<td>0.273</td>
<td>0.68</td>
</tr>
</tbody>
</table>
We will show mainly results obtained using the simple B&MII vertex, based on Experimental beta's.
C. Phonon exchange Induced Pairing Interaction (old theory)

*Barranco, Broglia, Gori, Vigezzi, Bortignon, Terasaki, PRL11 (1999)*

\[ M_{\nu',\lambda'}^{n} = \frac{\langle \nu' | R_{o} \frac{\partial U}{\partial r} Y_{\lambda} | \nu \rangle}{\sqrt{2j_{\nu} + 1}} \frac{\beta_{\lambda}(n)}{\sqrt{2\lambda + 1}} \]  \hspace{1cm} (1)

can be calculated. The quantities entering in the reduced matrix element appearing in Eq. (1) are the nuclear radius \( R_{o} \), the derivative of the Saxon-Woods potential, and a spherical harmonic of multipolarity \( \lambda \). Once these matrix elements are known, one can calculate the induced pairing interaction matrix elements (cf. inset of Fig. 1)

\[ \nu_{\nu'} = \langle (j_{\nu'} m_{\nu'}) (j_{\nu'} m_{\nu'}) | \nu \rangle (j_{\nu} m_{\nu}) (j_{\nu} m_{\nu}) \rangle_{a.s.} \]

\[ = \sum_{\lambda n} \frac{2}{(2j_{\nu} + 1)} \frac{2(M_{\nu',\lambda'}^{n})^{2}}{E_{o} - [e_{\nu} + e_{\nu'} + \hbar \omega_{\lambda}(n)]}, \]  \hspace{1cm} (2)

and thus determine the state dependent BCS pairing gap [2]

\[ \Delta_{\nu} = - \sum_{\nu'} \frac{(2j_{\nu} + 1)}{2} \frac{\Delta_{\nu'}}{2E_{\nu'}} \nu_{\nu'\nu}. \]  \hspace{1cm} (3)

\[ \text{IG. 1. State dependent pairing gap } \Delta_{\nu} \text{ [cf. Eq. (3)] for the nucleus } ^{120}\text{Sn}, \text{ calculated making use of the induced interaction defined in Eq. (2) (cf. inset, where particles are represented by arrowed lines and phonons by a wavy line).} \]
FIG. 1. The nucleus $^{120}\text{Sn}$. Diagonal pairing matrix elements of the induced interaction (upper panel, solid diamonds) and of the Gogny force (lower panel, solid circles), displayed as a function of the single-particle energy, $e_\nu$, of the state $\nu$ calculated using the bare nucleon mass and the single-particle wave functions of a Woods-Saxon potential with standard parameters (depth $V_0 = -49$ MeV, diffusivity $a = 0.65$ fm, and radius $R_0 = 6.16$ fm), including the spin-orbit term, parametrized according to Ref. [21]. Also shown by means of vertical lines is the position of the Fermi energy, $e_F = -9.1$ MeV. Note the different scale in the two figures.
Semiclassical (TF averaged) estimate of diagonal pairing matrix elements in $^{120}$Sn.

The coupling with the phonons induces a surface-peaked interaction and pairing gap

\[ \Delta \approx Z(v_{14} + v_{\text{ind}}) \kappa \]

But the pairing density \( k(r) \) is much less affected

Density modes (surface): Attractive
Spin modes (volume): Repulsive !!

Equivalent DDDI parameters **Volume Repulsive** Interaction!!
Induced Interaction in the Slab Model (Giovanardi, PRC65,041304(2002)):

D. Microscopic description of superfluid nuclei beyond mean field:

**QUASIPARTICLE VIBRATION COUPLING**

(cf. Van der Sluys et al., NPA551(1993)210)

For each single-particle state \( a = (q_a n_b l_a j_a) \) the equation-of-motion method leads to a system of linear equations of dimension \( 2(N + 1) \) with \( N \) the dimension of the \((1qp \times \text{phonon})\) space:

\[
\begin{pmatrix}
E_a & V(abJ\nu) & 0 & W(abJ\nu) \\
V(abJ\nu) (E_{J\nu} + E_b) & W(abJ\nu) & 0 & 0 \\
0 & W(abJ\nu) & -E_a & -V(abJ\nu) \\
W(abJ\nu) & 0 & -V(abJ\nu) & -(E_{J\nu} + E_b)
\end{pmatrix}
\begin{pmatrix}
x_a \\
x_b \\
C_{bJ\nu} \\
C_{bJ\nu}
\end{pmatrix}
= \tilde{E}_a
\begin{pmatrix}
x_0 \\
x_0 \\
y_0 \\
y_0
\end{pmatrix}.
\tag{23}
\]

Neglecting the mutual interaction between the \((1qp \times \text{phonon})\) states, it becomes easy to rewrite the secular equation (23) as a two-dimensional, hermitian and non-linear eigenvalue problem, by projecting onto the 1qp space:

\[
\begin{pmatrix}
E_a & 0 \\
0 & -E_a
\end{pmatrix} + \begin{pmatrix}
\Sigma_{11}(E) & \Sigma_{12}(E) \\
\Sigma_{12}(E) & \Sigma_{22}(E)
\end{pmatrix}
\begin{pmatrix}
x_0 \\
y_0
\end{pmatrix}
= \begin{pmatrix}
x_0 \\
y_0
\end{pmatrix}.
\tag{24}
\]

The energy-dependent self-energy matrix elements \( \Sigma_{ij}(E) \) stand for:

\[
\Sigma_{11}(E) = \sum_{bJ\nu} \left( \frac{|V(abJ\nu)|^2}{E - (E_b + E_{J\nu})} + \frac{|W(abJ\nu)|^2}{E + (E_b + E_{J\nu})} \right),
\]

\[
\Sigma_{22}(E) = \sum_{bJ\nu} \left( \frac{|W(abJ\nu)|^2}{E - (E_b + E_{J\nu})} + \frac{|V(abJ\nu)|^2}{E + (E_b + E_{J\nu})} \right),
\]

\[
\Sigma_{12}(E) = -\sum_{bJ\nu} V(abJ\nu) W(abJ\nu) \left( \frac{1}{E - (E_b + E_{J\nu})} - \frac{1}{E + (E_b + E_{J\nu})} \right).
\tag{25}
\]
USED FORMALISM
(Idini, Barranco, Vigezzi, PRC85, 014331(2012))

\[
\begin{bmatrix}
E_a + \Sigma_{11}(\tilde{E}_{a(n)}) & \Sigma_{12}(\tilde{E}_{a(n)}) \\
\Sigma_{12}(\tilde{E}_{a(n)}) & -E_a + \Sigma_{22}(\tilde{E}_{a(n)})
\end{bmatrix}
\begin{bmatrix}
x_{a(n)} \\
y_{a(n)}
\end{bmatrix} = \tilde{E}_{a(n)}
\begin{bmatrix}
x_{a(n)} \\
y_{a(n)}
\end{bmatrix}
\]

where one has introduced the normal and abnormal self-energies \(\Sigma_{11}(E)\) (being \(\Sigma_{22}(E) = -\Sigma_{11}(-E)\)) and \(\Sigma_{12}(E)\), given by

\[
\Sigma_{11} = \sum_{b,m,J,\nu} \frac{V^2(a(n)b(m)J\nu)}{\tilde{E}_{a(n)} - \tilde{E}_{b(m)} - \hbar\omega_{J,\nu}} + \sum_{b,m,J,\nu} \frac{W^2(a(n)b(m)J\nu)}{\tilde{E}_{a(n)} + \tilde{E}_{b(m)} + \hbar\omega_{J,\nu}}
\]

and

\[
\Sigma_{12} = \Sigma_{12}^{\text{pho}} + \Sigma_{12}^{\text{bare}}
\]

\[
\Sigma_{12}^{\text{pho}} = -\sum_{b,m,J,\nu} V(a(n),b(m),J,\nu)W(a(n),b(m),J,\nu)
\]

\[
\left[\frac{1}{\tilde{E}_{a(n)} - \tilde{E}_{b(m)} - \hbar\omega_{J,\nu}} - \frac{1}{E_{a(n)} + \tilde{E}_{b(m)} + \hbar\omega_{J,\nu}}\right].
\]

\[
\Sigma_{12}^{\text{bare}} = \pm \sum_{b,n} V_{\text{bare}}(a,b) \frac{(2j_b + 1)}{2} \tilde{u}_b(n) \tilde{v}_b(n).
\]

\[
V(jj'|\lambda) = \beta\lambda(2\lambda+1)^{-1/2} \langle j||\text{Ro} \frac{dU}{dr} \lambda ||j'\rangle (uj - vj'vj') (2j+1)^{-1/2}
\]

\[
W(jj'|\lambda) = \beta\lambda(2\lambda+1)^{-1/2} \langle j||\text{Ro} \frac{dU}{dr} \lambda ||j'\rangle (uj'v'j - vj'uj') (2j+1)^{-1/2}
\]

Optional: 1 or 2 steps?
Since self-energies are energy dependent many solutions are obtained: \( n=1,2,.. \)

Each carrying a quasi-particle strength \( u(a,n)^2 + v(a,n)^2 < 1 \)

Closure requires \( \Sigma_n u(a,n)^2 + v(a,n)^2 = 1 \)

\[
\begin{pmatrix}
(e_{a(n)}-e_F) + \Sigma_{11}(a, E_{a(n)}) \\
\Sigma_{12}(a, E_{a(n)})
\end{pmatrix}

\begin{pmatrix}
\Sigma_{12}(a, E_{a(n)}) \\
(e_{a(n)}-e_F) + \Sigma_{22}(a, E_{a(n)})
\end{pmatrix}

\begin{pmatrix}
u_{a(n)}
\end{pmatrix}

= E_{a(n)} \begin{pmatrix}
u_{a(n)}
\end{pmatrix}

\]

\[
Z_{a(n)} = \left(1 - \frac{\Sigma_{odd}(a, E_{a(n)})}{E_{a(n)}} \right)^{-1}
\]

\[
\begin{pmatrix}
(e_{a(n)}-e_F) \\
\Delta(a, E_{a(n)})
\end{pmatrix}

\begin{pmatrix}
\Delta(a, E_{a(n)}) \\
(e_{a(n)}-e_F)
\end{pmatrix}

\begin{pmatrix}
u_{a(n)}
\end{pmatrix}

= E_{a(n)} \begin{pmatrix}
u_{a(n)}
\end{pmatrix}
\]

\[
e_{a(n)}-e_F = Z_{a(n)} \left[ (\epsilon_a-e_F) + \Sigma_{even}(a, E_{a(n)}) \right]
\]

\[
\Delta_{a(n)} = Z_{a(n)} \left( \Sigma_{12}^{bare} + \Sigma_{12}^{pho} \right) = \frac{2 E_{a(n)} u_{a(n)} v_{a(n)}}{u_{a(n)}^2 + v_{a(n)}^2}
\]

where

\[
\Sigma_{odd}(a, E_{a(n)}) = \frac{\Sigma_{11}(a, E_{a(n)}) - \Sigma_{11}(a, -E_{a(n)})}{2}
\]

\[
\Sigma_{even}(a, E_{a(n)}) = \frac{\Sigma_{11}(a, E_{a(n)}) + \Sigma_{11}(a, -E_{a(n)})}{2}
\]
A generalized gap equation. Different versions

Expressing $u^*v$ as a function of sigma12, and reintroducing them in the sigma12 expression, a close expression for sigma is obtained:

$$\Delta_{a(n)} = -Z_{a(n)} \sum_{b(m)} V_{\text{eff}}(a(n), b(m)) N_{b(m)} \frac{\Sigma_{12}(b(m), E_{b(m)})}{2 \sqrt{(\epsilon_b - \epsilon_F + \Sigma_{\text{even}}(b(m), E_{b(m)}))^2 + \Sigma_{12}^2(b(m), E_{b(m)})}}$$

c.f. Baldo

where $N$ is the proper quasi-particle normalization:

$$N_{b(m)} = \hat{u}_{b(m)}^2 + \hat{v}_{b(m)}^2 = \left(1 - \frac{\partial \Sigma_{11}(a, E_{a(n)})}{\partial E_{a(n)}} u_{b(m)}^2 - \frac{\partial \Sigma_{22}(a, E_{a(n)})}{\partial E_{a(n)}} v_{b(m)}^2 - 2 \frac{\partial \Sigma_{12}(a, E_{a(n)})}{\partial E_{a(n)}} u_{b(m)} v_{b(m)} \right)^{-1} < 1$$

which gives the properly normalized $u,v$'s starting from the normalized to 1 $u,v$'s,

and where the effective interaction is

$$V_{\text{eff}}(a(n), b(m)) = V_{\text{bare}}(a, b) + \sum_{j,v} \hbar^2 (a,b,J,v) \left( \frac{1}{E_{an} - E_{bm} - \hbar \omega_{j,v}} - \frac{1}{E_{an} + E_{bm} + \hbar \omega_{j,v}} \right)$$

Reintroducing the $Z_b$-factor

$$\Delta_{a(n)} = -Z_{a(n)} \sum_{b(m)} V_{\text{eff}}(a(n), b(m)) N_{b(m)} \frac{Z_{b(m)} \Sigma_{12}(b(m), E_{b(m)})}{2 \sqrt{Z_{b(m)}^2 (\epsilon_b - \epsilon_F + \Sigma_{\text{even}}(b(m), E_{b(m)}))^2 + Z_{b(m)}^2 \Sigma_{12}^2(b(m), E_{b(m)})}}$$

$$\Delta_{a(n)} = -\sum_{b(m)} V_{\text{eff}}(a(n), b(m)) \frac{Z_{a(n)} \Delta_{b(m)} N_{b(m)}}{2 \sqrt{E_b - E_F}^2 + \Delta_{b(m)}^2}$$

$$\Delta_{a(n)} = -\sum_{b(m)} V_{\text{eff}}(a(n), b(m)) \frac{Z_{a(n)} \Delta_{b(m)} N_{b(m)}}{2 E_{b(m)}}$$
Z = 1  free Fermi gas
Z < 1  correlated Fermi system

Generalized Gap Equation (schematic)

\[ \Delta_{p} = - \frac{1}{2} \int d^{3}p' \left\{ \sum_{pp'} Z_{p} V_{pp'} Z_{p'} \right\} \sqrt{(\tilde{\epsilon}_{p'} - \epsilon_{F})^{2} + \Delta_{p'}^{2}} \]

Quasiparticle strength < 1

Bare + Induced interaction

Renormalized s.p. energy
RENORMALIZING  Argonne (on Sly4) pairing gaps (2 steps calculation)

Experimental phonons were used for the density modes $L=2,3,4,5$
Spin modes not included in this results
FIG. 17. (Color online) The state-dependent pairing gap $\Delta^{BCS}$ calculated in BCS with the bare $\rho_{14}$ interaction is compared to the renormalized gap $\bar{\Delta}$ [cf. Eq. (38)] obtained solving the Nambu-Gor'kov equations. We compare results obtained with a mean field produced with the SGII interaction (a) and with the SKa interaction (b) (cf. Fig. 8 for the corresponding calculation with the SLy4 mean field).

FIG. 18. The theoretical quasiparticle spectra obtained at the various steps of the calculation are compared to the experimental data. We compare results obtained with a mean field produced with the SGII interaction (a) and with the SKa interaction (b) (cf. Fig. 8 for the corresponding calculation with the SLy4 mean field).
Compare 1 and 2 step calculations
Using a simple monopole pairing force

FIG. 26. (Color online) Renormalized gaps $\tilde{\Delta}$ obtained solving the Nambu Gor'kov equations in the one-step and the two-steps diagonalization schemes with the monopole pairing force as a function of the pairing constant $G_0$, averaged over the five valence orbitals. Also shown is the gap $\Delta^{BCS}$ obtained solving the BCS equation.

 Allows correcting r-dependence

Barranco, Broglia, PRL 29(1987)

Application to 10Li and 11Be: Talk by E. Vigezzi
Fig. 1. Mean square radius of the Ca isotopes. In (a) the charge mean square radius is displayed as a function of the mass number [25–27]. The quantity $\langle r^2 \rangle^{1/2}_{\text{ch}}$ was assumed to have a linear dependence with $A$. Its slope was fitted by requiring that the quantity (1) coincides with the experimental values for $^{40}$Ca and $^{48}$Ca making use of the values of $\Delta r^2$ calculated through eqs. (2)–(4). These values are collected in table 1. The resulting values of $\langle r^2 \rangle_{\text{ch}}$ are displayed as a full drawn curve. In (b) the mass mean square radius is displayed [27]. The quantity $\langle r^2 \rangle^{1/2}_{\text{matt}}$ was given an $A^{1/3}$ dependence, and its coefficient adjusted to get the best fit when use is made of eq. (1) and the quantities $\Delta r^2$ (cf. table 1). The corresponding results are shown with a solid curve. The resulting bare mean square radius is $\langle r^2 \rangle^{1/2}_{\text{matt}} = A^{1/3} 0.96$ fm.
Figure A.2: Feynman representation of $V$ and $W$ vertices (A.2.8)-(A.2.11) for the case of particle $a$ represented in the quasiparticle basis (A.1.14)-(A.1.15) $(\pm)$ scattering into quasiparticle $b_{\mu}^{\pm}$ and a phonon $\lambda_{\nu}^{\pi}$.

\[ V(p_k^{(+)}, b_{\mu}^{(+)}, \lambda_{\nu}^{\pi}) \equiv \langle p_k^{(+)} | V_{res}^{\pi} | b_{\mu}^{(+)} \lambda_{\nu}^{\pi} \rangle = -\langle p_k^{(-)} | V_{res}^{\pi} | b_{\mu}^{(-)} \lambda_{\nu}^{\pi} \rangle = 0 | a_{p_k} \hat{V} \sum_l \tilde{u}_{b_{\mu},l}(a_{b_{\mu}}^{\dagger} + \tilde{v}_{b_{\nu},l} a_{b_{\nu}}) \Gamma \lambda_{\nu}^{\pi} | 0 \rangle \]

\[ = \sum_l (f + g)(p_k, b_{l}, \lambda_{\nu}^{\pi}) \tilde{u}_{b_{\mu},l} \quad \text{cf. Fig. A.2(a)} \quad (A.2.8) \]

\[ V(h_k^{(+)}, b_{\mu}^{(+)}, \lambda_{\nu}^{\pi}) \equiv \langle h_k^{(+)} | V_{res}^{\pi} | b_{\mu}^{(+)} \lambda_{\nu}^{\pi} \rangle = -\langle h_k^{(-)} | V_{res}^{\pi} | b_{\mu}^{(-)} \lambda_{\nu}^{\pi} \rangle = \sum_l -\tilde{v}_{b_{\mu},l}(f - g)(h_k, b_{l}, \lambda_{\nu}^{\pi}) \quad \text{cf. Fig. A.2(b)}, \quad (A.2.9) \]

\[ W(p_k^{(+)}, b_{\mu}^{(-)}, \lambda_{\nu}^{\pi}) \equiv \langle p_k^{(+)} | V_{res}^{\pi} | b_{\mu}^{(-)} \lambda_{\nu}^{\pi} \rangle = \langle p_k^{(-)} | V_{res}^{\pi} | b_{\mu}^{(+)} \lambda_{\nu}^{\pi} \rangle = \sum_l \tilde{v}_{b_{\mu},l}(f - g)(p_k, b_{l}, \lambda_{\nu}^{\pi}) \quad \text{cf. Fig. A.2(c)}, \quad (A.2.10) \]

\[ W(h_k^{(+)}, b_{\mu}^{(-)}, \lambda_{\nu}^{\pi}) \equiv \langle h_k^{(+)} | V_{res}^{\pi} | b_{\mu}^{(-)} \lambda_{n}^{\pi} \rangle = \langle h_k^{(-)} | V_{res}^{\pi} | b_{\mu}^{(+)} \lambda_{n}^{\pi} \rangle = \sum_l \tilde{u}_{b_{\mu},l}(f + g)(h_k, b_{l}, \lambda_{\nu}^{\pi}) \quad \text{cf. Fig. A.2(d)}. \quad (A.2.11) \]
Similar developments (Soma, et. al, PRC87, 011303(13))

Self-energy expansion

☆ Gorkov equations → energy **dependent** eigenvalue problem

\[
\sum_b \left( \frac{t_{ab} - \mu_{ab} + \Sigma_{ab}^{11}(\omega)}{\Sigma_{ab}^{21}(\omega)} - t_{ab} + \mu_{ab} + \Sigma_{ab}^{22}(\omega) \right) \left|_{\omega_k} \begin{pmatrix} \mathcal{U}_b^k \\ \mathcal{V}_b^k \end{pmatrix} \right. = \omega_k \left( \begin{pmatrix} \mathcal{U}_a^k \\ \mathcal{V}_a^k \end{pmatrix} \right)
\]

Ab Initio, and HF side, 2\textsuperscript{nd} order

☆ 1\textsuperscript{st} order → energy-**independent** self-energy

☆ 2\textsuperscript{nd} order → energy-**dependent** self-energy
Scaling of Gorkov’s problem

★ Transformed into an energy independent eigenvalue problem

\[
\begin{pmatrix}
T - \mu + \Lambda & \tilde{h} & C & -D^\dagger \\
\tilde{h}^\dagger & -T + \mu - \Lambda & -D^\dagger & C \\
C^\dagger & -D & E & 0 \\
-D & C^\dagger & 0 & -E
\end{pmatrix}
\begin{pmatrix}
U^k \\
V^k \\
W_k \\
Z_k
\end{pmatrix}
= \omega_k
\begin{pmatrix}
U^k \\
V^k \\
W_k \\
Z_k
\end{pmatrix}
\]

★ Numerical scaling

\[m_{p,1} \approx \left( \frac{N_b}{3} \right) \propto \frac{N_b^3}{6}\]

\[
2N_b \begin{Bmatrix}
N_b \\
N_b \\
N_{tot}
\end{Bmatrix}
\]

- \(N_b\) \rightarrow \text{dimension of the s.p. basis}
- \(n\) \rightarrow \text{number of iterations}

\[N_{tot,1} = 2N_b + M_{p,1} \approx N_b^3\]
\[N_{tot,n} = 2N_b + M_{p,n} \approx N_b^{3n}\]
Pairing gaps

★ Three-point mass differences

$$\Delta_n^{(3)}(A) = \frac{(-1)^A}{2} [E_0^{A+1} - 2E_0^A + E_0^{A-1}]$$
FIG. 3. (Color online) Three-point mass differences $\Delta^{(3)}_n$ in the calcium isotopes calculated to third order in MBPT with and without the leading chiral 3N forces, and in comparison with experiment [24, 67]. The legend is as in Fig. 1. Panel (a) shows the results of the third-order ladder contributions. Panels (b) and (c) include all MBPT diagrams to third order in the $pf$-shell and the extended $pgf_{9/2}$ valence space, respectively. The results in the $pf$-shell are with empirical SPEs. For the $pgf_{9/2}$ space, we show pairing gaps for both the MBPT and empirical SPEs.

When particle-hole contributions are included in a full third-order calculation, we find in Fig. 3 a clear improvement compared to including only ladder diagrams. In the $pf$-shell, the three-point mass differences are increased, leading to reasonable agreement with experimental data. This clearly demonstrates the importance of particle-hole many-body processes, such as core-polarization, on pairing in nuclei. Our results show that they can provide the missing pairing strength required to reproduce experiment on top of the direct NN+3N interactions. Analogously, the systematic differences between theoretical and experimental pairing gaps found in the EDF approach of Ref. [15] may be attributed to these effects.
C. BARBIERI AND B. K. JENNINGS PRC72,014613(2005)

FIG. 1. (Color online) Feynman diagrams representation of the self energy. The first diagram on the right-hand side represents the Hartree-Fock–like contribution to the mean field. The remaining ones describe core polarization effects in the particle (2p1h) and hole (2h1p) part of the spectrum.

Combines i)particle, ii)ph-RPA and iii)pp(hh)-RPA consistently

FIG. 2. (Color online) Example of a diagrammatic contribution included in the Faddeev expansion for $R^{(2p1h)}$ (see Fig. 1). A quasiparticle is coupled to the response function $\Pi^{(ph)}$ that describes the target nucleus. It can also participate in pairing processes, which are accounted for by the two-body propagator $g^{II,pp}$. 

Fadeev RPA, Barbieri, Dickhoff; PRC63, 034313(2001)
CONCLUSIONS

a. At mean field level surface pairing interaction preferred. 
   Bare Argonne tractable as pairing interaction.

b. Phonon Exchange Pairing Induced Interaction very 
   surface peaked. In volume it could be repulsive due 
   to spin modes. 
   Strong A-dependence from the Slab Model.

c. Close connection to self-energy effects from 
   Dyson Gorkov's eqs.: Quasi-particle Phonon Coupling: 
   Fragmentation and Dense Eigenstates<=> Induced Pairing Int. 
   Bare Argonne plus Induced Pairing good description of data.

d. Shell Model confirms relevance of surface phonons mediated 
   pairing interaction.

e. Multishell Self-energy: Full HFB renormalization. 
   Ab Initio calculations in progress (2nd order).