\( \eta \) and \( \eta' \) mesons
from \( N_f = 2 + 1 + 1 \) flavour lattice QCD
for the ETM collaboration

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Flavour Singlet Pseudo-Scalar Mesons

- nine lightest pseudo-scalar mesons show a peculiar spectrum:
  - 3 very light pions (140 MeV)
  - kaons and the $\eta$ around 600 MeV
  - $\eta'$ around 1 GeV

- The large mass of the $\eta'$ meson is thought to be caused by the QCD vacuum structure and the $U_A(1)$ anomaly

- $\eta'$ meson is not a (would be) Goldstone Boson

$\Rightarrow$ massive even in the SU(3) chiral limit
Lattice Status

- disconnected contributions significant
  ⇒ hard problem

- only a limited amount of lattice results available

- no control of systematics
  - usually only one lattice spacing
  - and/or only one pion mass

  ⇒ no clear picture

- in particular at light pion masses

$N_f = 2 + 1 + 1$ Wilson Twisted Mass Fermions

- with twisted mass formulation of LQCD only *doublets* of quarks can be considered

- light doublet, mass-degenerate:
  
  \[ D_\ell = D_W + m_{\text{crit}} + i\mu_\ell \gamma_5 \tau^3, \quad \chi_\ell = \begin{pmatrix} \chi_u \\ \chi_d \end{pmatrix} \]

- heavy doublet, mass-split, flavour non-diagonal:
  
  \[ D_h = D_W + m_{\text{crit}} + i\mu_\sigma \gamma_5 \tau^1 + \mu_\delta \tau^3, \quad \chi_h = \begin{pmatrix} \chi_c \\ \chi_s \end{pmatrix} \]

[Frezzotti, Rossi (2004)]

- rotation from $\chi$- to standard $\psi$-basis (at maximal twist):
  
  \[ \psi_\ell = e^{i\pi \gamma_5 \tau^3/4} \chi_\ell, \quad \psi_h = e^{i\pi \gamma_5 \tau^1/4} \chi_h \]
$N_f = 2 + 1 + 1$ Wilson Twisted Mass Fermions

Pros:
- $O(a)$ improvement at maximal twist
  
  [Frezzotti, Rossi; JHEP 0408 (2004)]

  ⇐ by tuning only one parameter

  - excellent scaling behaviour observed

  - mixing patterns under renormalisation can be simplified

  - note: could easily introduce $u$-$d$ mass splitting as well

Cons:
- flavour and parity symmetries broken at finite values of the lattice spacing
  ⇒ technical complication

  ⇒ unphysical splittings, mostly in between $m_{\pi \pm}$ and $m_{\pi^0}$

  [Dimopoulos, Frezzotti, Michael, Rossi, CU; Phys.Rev. D81 (2010)]
The 1 + 1 Doublet

- heavy doublet:

\[ D_h = D_W + m_{\text{crit}} + i\mu_\sigma\gamma_5\tau^1 + \mu_\delta\tau^3, \quad \chi_h = \begin{pmatrix} \chi_c \\ \chi_s \end{pmatrix} \]

- think of \( \mu_\sigma \) as the mean s/c mass and \( \mu_\delta \) the splitting

- splitting \( \mu_\delta \) orthogonal to twist \( \mu_\sigma \)

(\( \tau^3 \) versus \( \tau^1 \))

- obtain renormalised quark masses of the doublet

\[ \hat{m}_s = Z_p^{-1}\mu_\sigma - Z_s^{-1}\mu_\delta \]
\[ \hat{m}_c = Z_p^{-1}\mu_\sigma + Z_s^{-1}\mu_\delta \]

- fermion determinant positive and \( O(a) \) improvement remains valid

[Frezzotti, Rossi (2004)]
Flavour Singlet Pseudo-Scalar Mesons

• in the SU(3) symmetric case (sloppy notation)

\[ \eta_8 : \bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d - 2\bar{s}i\gamma_5 s \]

\[ \eta_0 : \bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d + \bar{s}i\gamma_5 s \]

• SU(3) symmetry broken \(\Rightarrow\) mixing

\(\Rightarrow\) SU(2) plus strange:

\[
\begin{pmatrix}
|\eta\rangle \\
|\eta'\rangle
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix} \cdot
\begin{pmatrix}
|\eta_\ell\rangle \\
|\eta_s\rangle
\end{pmatrix}
\]

• \(N_f = 2 + 1 + 1\) possible charm contribution


\(\eta, \eta'\) from LQCD
Flavour Singlet Pseudo-Scalar Mesons

- need to estimate correlator matrix

\[ C = \begin{pmatrix}
\eta_{ll} & \eta_{ls} & \eta_{lc} \\
\eta_{sl} & \eta_{ss} & \eta_{sc} \\
\eta_{cl} & \eta_{cs} & \eta_{cc}
\end{pmatrix} \]

- \( \eta_{XY} \) correlator of appropriate interpolating fields, e.g.

\[ \eta_{ss}(t) \equiv \langle \bar{s}i\gamma_5 s(t) \bar{s}i\gamma_5 s(0) \rangle \]

projected to zero momentum

- \( \eta \): lowest state, \( \eta' \): first state, \( \eta_c \) ...
A Little Twisted Mass Algebra

- rotation to twisted basis

\[
\frac{1}{\sqrt{2}} (\bar{\psi}_u i \gamma_5 \psi_u + \bar{\psi}_d i \gamma_5 \psi_d) \rightarrow \frac{1}{\sqrt{2}} (\bar{\chi}_d \chi_d - \bar{\chi}_u \chi_u) \equiv \mathcal{O}_\ell ,
\]

- and in the heavy sector

\[
\begin{pmatrix} \bar{\psi}_c \\ \bar{\psi}_s \end{pmatrix}^T i \gamma_5 \frac{1 \pm \tau^3}{2} \begin{pmatrix} \psi_c \\ \psi_s \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\chi}_c \\ \bar{\chi}_s \end{pmatrix}^T -\tau^1 \pm i \gamma_5 \tau^3 \frac{1}{2} \begin{pmatrix} \chi_c \\ \chi_s \end{pmatrix} \equiv \mathcal{O}_{c,s} .
\]

- therefore

\[
\mathcal{O}_c \equiv Z (\bar{\chi}_c i \gamma_5 \chi_c - \bar{\chi}_s i \gamma_5 \chi_s)/2 - (\bar{\chi}_s \chi_c + \bar{\chi}_c \chi_s)/2 ,
\]

\[
\mathcal{O}_s \equiv Z (\bar{\chi}_s i \gamma_5 \chi_s - \bar{\chi}_c i \gamma_5 \chi_c)/2 - (\bar{\chi}_s \chi_c + \bar{\chi}_c \chi_s)/2 .
\]

- with ratio of non-singlet renormalisation constants

\[
Z \equiv \frac{Z_P}{Z_S}
\]
Estimating Disconnected Diagrams

- fermionic disconnected contributions noisy
  \[ \Rightarrow \text{need as many as possible observations} \]

- use \( R \) stochastic volume sources \( \xi^r \)
  \[
  \lim_{R \to \infty} [\xi_i^* \xi_j]_R = \delta_{ij}, \quad \lim_{R \to \infty} [\xi_i \xi_j]_R = 0
  \]

- then we get
  \[
  [\xi_i^r \phi_j] = (D^{-1})_{ji} + \text{noise}
  \]
  with
  \[
  \phi_j = (D^{-1})_{jk} \xi_k^r
  \]

- noise \( \propto \sqrt{V_s/R} \) while signal \( \mathcal{O}(1) \)
A Very Efficient Variance Reduction Method


- relation in between up and down Dirac operator
  \[ D_u - D_d = 2 \mu_\ell \gamma_5 \]
- multiply with \( 1/D_u \) from left, \( 1/D_d \) from right
  \[ \frac{1}{D_d} - \frac{1}{D_u} = 2 \mu_\ell \frac{1}{D_u} \gamma_5 \frac{1}{D_d} \]
- Lhs is what we want, rhs has an extra volume loop
  \Rightarrow \text{can be estimated from}
  \[ [\phi^* \phi]_R + \text{noise} \]
- noise \( \propto \sqrt{V_s^2/R} \), but signal \( \propto V \)
- only applicable in the light sector involving \( \tau^3 \)
A Very Efficient Variance Reduction Method: Example

- Example for strangeness content of the nucleon
  [ETMC, Dinter et al., JHEP 1208 (2012)]

- Easily a factor four improvement

\[ N_R(t_0=6) dR(t_0=6) \]
Ensemble-Details

- gauge configurations from ETM Collaboration
  [ETMC, R. Baron et. al., JHEP 06 111 (2010)]

- Iwasaki Gauge action
  [Iwasaki, Nucl. Phys. B258, 141]

- three lattice spacings:
  \[ a_A = 0.086 \text{ fm}, \quad a_B = 0.078 \text{ fm} \quad \text{and} \quad a_D = 0.061 \text{ fm} \]

- charged pion masses range from \( \approx 230 \text{ MeV} \) to \( \approx 500 \text{ MeV} \)

- \( L \geq 3 \text{ fm} \) and \( M_\pi \cdot L \geq 3.5 \) for most ensembles

- \( \approx 600 \) up to \( \approx 2500 \) gauge configuration per ensemble

- \( \mu_\sigma, \mu_\delta \) fixed for each \( \beta \)

- use \( r_0 = 0.45(2) \text{ fm} \) (from \( f_\pi \)) throughout the talk
Analysis Procedure

- 24 to 32 volume sources per gauge for disconnecteds have tested one ensemble with 64 sources
- local and smeared operators
- two $\gamma$-combinations $i\gamma_5$, $i\gamma_0\gamma_5$
- two independent fitting methods (up to $12 \times 12$ matrix)
  - solving the GEVP
    - using a factorising fit
- errors computed by bootstrapping and blocking 1000 bootstrap samples
  ⇒ significant autocorrelation for $\eta'$: $\sim 20$ trajectories
Effective Masses $B_{25}$

3 × 3 matrix

- ground state $\eta$ well determined
- next state hardly plateaus

6 × 6 matrix

$\eta, \eta'$ from LQCD
flavour content qualitatively as expected
- no charm contribution to $\eta$ and $\eta'$
- third state (not shown) is charm only (almost)
Summary Masses

- $\eta$ mass quite precise
- $\eta'$ rather noisy
  large systematic uncertainties
- mild pion mass dependence in $M_\eta$
- physical strange quark mass not fixed for different $\beta$ values!
- what can we say about lattice artifacts?
- how to perform the chiral extrapolation?
Strange Quark Mass Dependence

- \( \mu_\sigma \) and \( \mu_\delta \) fixed for each \( \beta \)

- \( m_s \) unfortunately not perfectly tuned to its physical value

- we have two re-tuned ensembles for \( a_A (\beta = 1.90) \)

\( \Rightarrow \) can estimate \( m_s \) dependence

\( \Rightarrow \) need to come up with a strategy to correct for this effect!
Scaling Test for $M_\eta$ \hspace{1cm} (1)

⇒ fix $r_0M_{PS}$, $r_0M_K$ and $V/r_0$

• we don’t see a volume dependence in $M_\eta$
⇒ ignore it

• estimate

\[ D_\eta \equiv \frac{d(aM_\eta)^2}{d(aM_K)^2} = 1.6(2) \]

from two $A$-ensembles

• now assume:
$D_\eta$ independent of $\beta, \mu_\ell, \mu_\sigma, \mu_\delta$
Scaling Test for $M_\eta$ (2)

- use ensembles $A60$, $B55$, $D45$ with $r_0 M_{PS} \approx 0.9$

- correct $M_\eta$ using $D_\eta$ linearly in $M_K^2$

  $\Rightarrow r_0 M_K = 1.34$ fixed

- compatible with both, constant and linear continuum extrapolation

  $\Rightarrow$ assign conservative 8% error from difference to all our results
Chiral Extrapolation of $M_\eta$

- more ambitious: shift all $M_\eta$ to physical strange mass
- fit
  \[ g_K = a + b(r_0 M_{PS})^2 \]
  to data for $(r_0 M_K)^2$ from $A$ ensembles
- adjust $a$ to match physical $M_K$ for $M_{PS} = M_\pi \Rightarrow \tilde{g}_K$
- compute
  \[ \delta_K[(r_0 M_{PS})^2] = (r_0 M_K)^2 - \tilde{g}_K[(r_0 M_{PS})^2] \]
  for all ensembles
Chiral Extrapolation of $M_\eta$

- now correct all $(r_0 M_\eta)^2$ by corresponding

\[ D_\eta \cdot \delta_K [(r_0 M_{PS})^2] \]

\[ \Rightarrow (r_0 \bar{M}_\eta)^2 [(r_0 M_{PS})^2] \]

- all $\beta$-values fall on the same curve!

- extrapolate $(r_0 \bar{M}_\eta)^2$ linearly in $(r_0 M_{PS})^2$ to $M_{PS} = M_\pi$

- result

\[ M_\eta = 549(33)_{\text{stat}}(44)_{\text{sys}} \text{ MeV} \]
Chiral Extrapolation of $M_\eta$

- alternatives to avoid $D_\eta$
  - use the ratio $(M_\eta/M_K)^2$

- all $\beta$-values on a single curve
  - in particular: $m_s$ dependence seems to cancel

- extrapolate linearly in $(r_0M_{PS})^2$ to physical point

- result

  $$M_\eta = 558(13)_{\text{stat}}(45)_{\text{sys}} \text{ MeV}$$
Chiral Extrapolation of $M_\eta$

- or use GMO relation

\[ 3M_\eta^2 = 4M_K^2 - M_\pi^2 \]

valid for SU(3)

- experimentally $\sim 0.925$

- result

\[ M_\eta = 559(14)_{\text{stat}}(45)_{\text{sys}} \text{ MeV} \]

- weighted average over three methods

\[ M_\eta = 557(15)_{\text{stat}}(45)_{\text{sys}} \text{ MeV} \]
Comparing to Other Lattice Results

- overall mutual agreement

- apart from maybe UKQCD at smallest $M_{PS}$

- we have pushed significantly more chiral

- ETMCs $\eta'$ is much noisier despite
  - similar number of independent gauges (RBC, HSC, ETMC)
  - similar number of inversions (RBC, ETMC)
η and η’ Mixing

- write correlator matrix

\[ C_{qq'}(t) = \sum_n A_{q,n} A_{q',n} \frac{1}{2m(n)} \left[ \exp(-m(n)t) + \exp(-m(n)(T-t)) \right] \]

with amplitudes \( A_{q,n} \) corresponding to \( \langle 0|\bar{q}q|n\rangle \) (\( n \equiv \eta, \eta', \ldots \) and \( q = \ell, s, c \))

- define mixing angles via (ignoring charm)

\[
\begin{pmatrix} A_{\ell,\eta} & A_{s,\eta} \\ A_{\ell,\eta'} & A_{s,\eta'} \end{pmatrix} = \begin{pmatrix} f_\ell \cos \phi_\ell & -f_s \sin \phi_s \\ f_\ell \sin \phi_\ell & f_s \cos \phi_s \end{pmatrix}
\]

- for \( \phi_s \approx \phi_\ell \)

\[
\tan^2 \phi = -\frac{A_{\ell,\eta'} A_{s,\eta}}{A_{\ell,\eta} A_{s,\eta'}}
\]
\(\eta\) and \(\eta'\) Mixing

- \(\phi_\ell\) and \(\phi_s\) are too noisy separately

- single mixing angle \(\phi\) can be determined

- linear fit in \((r_0 M_{PS})^2\)
  \[
  \phi = 44(5)^\circ
  \]

  or in singlet/octet basis
  \[
  \theta = -10(5)^\circ
  \]

\(\Rightarrow\) good agreement with other determinations

- note: statistical error only
Mixed Action Approach

• idea: use different valence action for $s$ and $c$
  ⇒ to avoid $s/c$ flavour mixing
  ⇒ to vary strange and charm quark masses

• up/down stay unitary

• add two valence strange quarks $s$ and $s'$

\[
s \equiv s(+) : D_W + m_{\text{crit}} + \mu_s i\gamma_5
\]

\[
s' \equiv s(-) : D_W + m_{\text{crit}} - \mu_s i\gamma_5
\]

• can proof that continuum limit is correct
  [Frezzotti, Rossi, JHEP 08, 007 (2004)]

• the different signs allow to use the variance reduction trick

\[
\bar{\psi}_s \psi_s = \frac{1}{2} (\bar{\psi}_s \psi_s + \bar{\psi}_s \psi_s) = \frac{1}{2} (\bar{\chi}_s \chi_s - \bar{\chi}_s' \chi_s')
\]
Matching Unitary and Mixed Actions

- there are different matching quantities possible
  - match $\mu_\delta$ with $\mu_\sigma - Z\mu_\delta$
  - unitary with mixed $\bar{s}(+d(-)$ kaon (denote $M_{K^+}$)
    $\Rightarrow$ smallest lattice artifacts for $f_{PS}$ and $M_{PS}$
    [Sharpe, Wu, Phys. Rev. D 71 (2005), Frezzotti et al., JHEP 06 04 (2006)]
  - unitary with mixed $\bar{s}(-d(-)$ kaon (denote $M_{K^0S}$)
    $\Rightarrow$ usually larger lattice artifacts
  - and of course many other quantities

$\Rightarrow$ tried first to match with $M_{K^+}$
Matching Unitary and Mixed Actions

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  - match $\mu_\delta$ with $\mu_\sigma - Z\mu_\delta$
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    $\Rightarrow$ usually larger lattice artifacts
  - and of course many other quantities
  $\Rightarrow$ tried first to match with $M_{K^+}$

- ... and got pathological results

- so, why is that?
Matching Unitary and Mixed Actions

- $\eta, \eta'$ have significant disconnected contributions
  $\Rightarrow$ which don't know about $\mu_s$!

- connected and disconnected have to match to produce the correct correlation matrix
  $\Rightarrow$ match connected only $M_{\eta ss}$

- due to $M_{PS^\pm} - M_{PS^0}$ splitting: significantly smaller matching $\mu_s$-value
$\mu_S$-Dependence

\[ m_{\text{eff}} \cdot r_0 \]

\[ \mu S \cdot r_0 \]

- at $M_{\eta_{ss}}$ matching point we find reasonable agreement in between unitary and mixed approach
  - currently exploring this further
Summary

- $\eta$ and $\eta'$ for three lattice spacings and various quark mass values
- $\eta$ can be extracted precisely
  
  \[ M_\eta = 557(15)_{\text{stat}}(45)_{\text{sys}} \text{ MeV} \]
- $\eta'$ noisy, significant systematics
- Single mixing angle
  
  \[ \phi = 44(5)^\circ \text{ or } \theta = 10(5)^\circ \]
- Small lattice artifacts in $M_\eta$
Outlook

- noise reduction techniques for $\eta'$
- larger operator basis
- $\eta\pi$ scattering length
- $\eta \rightarrow \gamma\gamma$
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$\phi_\ell$ and $\phi_s$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Plot showing the dependence of $\phi_\ell$ and $\phi_s$ on $(r_0 M_{PS})^2$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Plot showing the dependence of $\phi_\ell$ and $\phi_s$ on $(r_0 M_{PS})^2$.}
\end{figure}