Quantum simulating real-time dynamics in gauge theories

Collaborators:
Pascal Stebler, Philippe Widmer, Uwe-Jens Wiese (Bern)
Markus Müller (Madrid)
Enrique Rico-Ortega, Marcello Dalmonte, Peter Zoller (Innsbruck)

Debasish Banerjee

Institute for Theoretical Physics, Bern, Switzerland

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**Sign Problem in it’s various incarnations**

- **Finite density**: finite $\mu_B$ QCD prime example. Simpler examples (e.g. scalar theories) can be constructed $\Rightarrow$ Complex actions $\Rightarrow$ certain cases can be solved with reformulations (world line, worm) or complex Langevin.

- **Fermions**: repulsive fermionic Hubbard model away from half filling prime example. Meron cluster algorithms can be used to solve certain sign problems.

- **Non-zero $\theta$ angle**: QCD at non-zero $\theta$. Again, simpler models at finite-$\theta$ (e.g. 2d O(3), CP(N-1)) can be solved with meron cluster methods.

- **Geometrically frustrated anti-ferromagnets**: Meron cluster method can again help in certain cases.

- **Real time evolution**: No method known to solve any simple non-trivial model at sufficiently large lattices. *Quantum Simulation with cold atoms/molecules and/or Rydberg ions*.

In it’s most general form **sign problem is NP hard** Troyer, Wiese (2005). General solution applicable to all problems unlikely.
Quantum Simulation: Analogue

Basic idea: system of interest → model Hamiltonian (usually Hubbard type) → implement via optical lattices

cold atoms in optical lattices realize Bosonic and Fermionic Hubbard models

$$H = -\kappa(t) \sum_{i,j} (b_i^\dagger b_j + b_j^\dagger b_i) + U(t) \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

Example: Observation of Mott-insulator (disordered) to superfluid (ordered) phase. Excitation spectrum probed. Greiner et. al. (2002)
Quantum Simulation: Digital

Ions confined in an ion-trap; but interactions between individual ions can be controlled using gates. Engineering Hamiltonian not required. More control over interactions; complicated interactions can be programmed in; Main challenge: scalability to large systems

Small-scale prototype of quantum computer
An example of real-time evolution

Use the Trotter-Suzuki decomposition

\[ e^{-iHt} \approx e^{-iH_1 t} e^{-iH_2 t} e^{[H_1, H_2]t^2/2} \]

to study the real time evolution of 2-Ising spins

Time-dependent variation of parameters possible
Trotter errors known and bounded; gate errors under control;
Implementation with up to 6 ions/spins  
Lanyon et. al. 2011
What to see in *real* time?

Confinement in QCD is phenomenologically explained by a “string”

\[ V(r) = A + \frac{B}{r} + \sigma r \]

G.S. Bali, K. Schilling (1992); Bali et. al. (2005).
What to see in real time?

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What to see in *real* time?

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The importance of being ’Quantum’

- Wilson formulation has continuum valued fields at each link and site; Unsuitable for AMO realisation
- Quantum Link Models (QLM) Chandrasekharan Wiese (1996) reformulate LGT to have discrete Hilbert spaces at each link, but generate continuum valued GT
- Example: the U(1) Quantum Link model
- Recall Wilson’s formulation:

  \[ u_{x,\mu} = e^{i\phi_{x,\mu}} = \cos(\phi_{x,\mu}) + i\sin(\phi_{x,\mu}) \]

  \[ S = -\frac{J}{2} \sum_{x,\mu > \nu} (u_{x,\mu} u_{x+\mu,\nu} u_{x+\nu,\mu}^\dagger u_{x,\nu}^\dagger + h.c) \]

  Classical action \(\rightarrow\) simulating a classical stat. mech. system

- Quantum counterpart \(\rightarrow\) quantum H operator

  \[ S = -\frac{J}{2} \sum_{x,\mu > \nu} (U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^\dagger U_{x,\nu}^\dagger + h.c) \]

  \(U_{x,\mu}\): operator acting on a discrete Hilbert space
The U(1) Quantum Link model

- Gauss Law generates gauge transformations. \([H, G_x] = 0:\)

\[
U'_x,\mu = \left[ \prod_m \exp(-i\alpha_m G_m) \right] U_{x,\mu} \left[ \prod_n \exp(i\alpha_n G_n) \right] = \exp(i\alpha_x) U_{x,\mu}
\]

Commutation relations \([G_x, U_{y,\mu}] = (\delta_{x,y+\mu} - \delta_{x,y}) U_{y,\mu}\) and \([G_x, U_{y,\mu}^\dagger] = (\delta_{x,y} - \delta_{x,y+\mu}) U_{y,\mu}^\dagger\) ensures Gauge invariance

- GT generated by electric fields: \(G_x = \sum_\mu (E_{x,\mu} - E_{x-\mu,x})\)

- Consider the spin-1/2 representation. Then, \(E_{x,\mu} = \sigma^z\)

  Eigenvalues: \(\pm \frac{1}{2}\)

- Pictorially:

  ![Diagram](image)

- Gauss’ Law: \(G_x |\psi\rangle = 0\)
The U(1) QLM: Hamiltonian

- In the spin-1/2 representation, \( U = S^+; U^\dagger = S^- \)
- The Hamiltonian acts by flipping a plaquette:

\[
H \begin{array}{c}
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\end{array} = J \begin{array}{c}
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\end{array}
\]

\[
H \begin{array}{c}
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
\end{array} = 0
\]

- Related to the Rokshar-Kivelson model, a candidate for exhibiting the spin-liquid phase

\[
H_{RK} = H_{QLM} - \lambda \sum_x (\delta_{\text{flippable}})
\]
The U(1) QLM: Spectrum
The U(1) QLM: Confinement
Schwinger model

Stick to simple models in 1-d having the same qualitative features for validating the quantum simulator. Schwinger model with QL in spin-1 representation and staggered fermions:

\[
H = -\sum_x \left[ \psi_x^\dagger U_{x,x+1} \psi_{x+1} + \text{h.c.} \right] + m \sum_x (-1)^x \psi_x^\dagger \psi_x + \frac{g^2}{2} \sum_x E_{x,x+1}^2 - 2\lambda \sum_x (-1)^x E_{x,x+1}
\]

Translation symmetry by even number of lattice spacings
Gauss’ Law: \([H, G_x] = 0\); \(G_x = \psi_x^\dagger \psi_x + \frac{(-1)^x - 1}{2} - (E_{x,x+1} - E_{x-1,x})\)
Parity: \(P \psi_x \rightarrow \psi_{-x}, \ P \psi_x^\dagger \rightarrow \psi_{-x}^\dagger, \ P U_{x,x+1} \rightarrow U_{-x-1,-x}^\dagger, \ P E_{x,x+1} \rightarrow -E_{-x-1,-x}\)
Charge Conjugation: \(C \psi_x \rightarrow (-1)^{x+1} \psi_{x+1}^\dagger, \ C \psi_x^\dagger \rightarrow (-1)^{x+1} \psi_{x+1}, \ C U_{x,x+1} \rightarrow U_{x+1,x+2}^\dagger, \ C E_{x,x+1} \rightarrow -E_{x+1,x+2}\)
Discrete Chiral Symmetry: broken by the mass term
\(x \psi_x \rightarrow \psi_{x+1}, \ x \psi_x^\dagger \rightarrow \psi_{x+1}^\dagger, \ x U_{x,x+1} \rightarrow U_{x+1,x+2}, \ x E_{x,x+1} \rightarrow E_{x+1,x+2}\)
The String and its breaking

\[ a) \]

\[ \frac{1}{2} \]

\[ \bar{Q} \]

\[ b) \]

\[ \bar{Q} \quad q \quad \bar{q} \quad Q \]

\[ c) \]

\[ Q \]

\[ d) \]

\[ \bar{Q} \quad q \quad \bar{q} \quad q \quad \bar{q} \quad Q \]

\[ e) \]

\[ \bar{Q} \quad q \quad \bar{q} \quad Q \]

\[ \text{meson} \quad \text{vacuum} \quad \text{meson} \]
Energetics

- Energetics at $t \to 0$ straightforward to calculate. The $\lambda$ term is redundant. String formation and breaking involves the gauge coupling.
- Vacuum state: $E_0 = -mL^2$
- Energy for the unbroken string state: $E_{\text{string}} - E_0 = \frac{g^2}{2} (L - 1)$
- Energy for the two meson state: $E_{\text{mesons}} - E_0 = 2\left(\frac{g^2}{2} + m\right)$
- Energy difference: $E_{\text{string}} - E_{\text{mesons}} = \frac{g^2}{2} (L - 3) - 2m$
- Critical distance for string breaking obtained from:

$$E_{\text{string}} - E_{\text{mesons}} = 0 \implies L = \frac{4m}{g^2} + 3$$

- Sites and links next to the two boundaries do not change during the whole evolution. Quantum simulation of system size $L$ needs $L + (L - 1) - 4 = 2L - 5$ ions
Static Properties

$E_{\overline{QQ}} - E_{00;0}$ vs $L$

- $m = 0$
- $m = 0.5$
- $m = 1$
- $m = 1.5$
- $m = 2$
- $m = 2.5$
- $m = 3$
Dotted dashed: $\langle 0 | \sum_x E_{x,x+1} | 0 \rangle$; Dashed: $\langle br(t) | \sum_x E_{x,x+1} | br(t) \rangle$; Continuous: $\langle unbr(t) | \sum_x E_{x,x+1} | unbr(t) \rangle$
False Vacuum Decay

- Consider the spin-1/2 representation. Then, the $g^2$ term trivial.
- Useful to consider the term: $-2\lambda \sum_x (-1)^x E_{x,x+1}$.
- $C, P$ exact symmetries with the $C \& P$ inv ground state for $m < \lambda (\lambda > 0)$.

- $m < \lambda$: two competing ground states exist, $CP$ partners of each other.

- Mimics $\theta = \pi$ in Schwinger model.
- Decay of false $C$ and $P$ invariant vacuum by bubble nucleation into true vacuum with spontaneous breaking of $C$ and $P$ is another interesting exercise in real-time evolution of a quantum system, that cannot be done by classical computers.
Microscopic and Effective theory

- The Schwinger model acts as an effective theory denote it by \( \mathcal{H} = \{ \mathcal{H}_F, \mathcal{H}_S \} \) induced at low-energies by a microscopic Hubbard type model denoted by \( H = \{ H_F, H_S \} = H_{ph} \oplus H_{unph} \).

- Idea: Clear separation of energy scales between \( H_{ph} \) and \( H_{unph} \).

- \( \mathcal{H} \) should satisfy Gauss Law while acting in \( H_{ph} \). States in \( H_{unph} \) need not.

- One-to-one correspondence between states in \( \mathcal{H} \) and the physical Hilbert space of \( H \), i.e.,

\[
\mathcal{H}|\Psi\rangle = \epsilon_\Psi|\Psi'\rangle \Rightarrow H|\Phi\rangle = \epsilon_\Phi|\Phi'\rangle \forall |\Phi\rangle \in H_{ph}
\]

- \( \epsilon_\Psi \) and \( \epsilon_\Phi \) same upto shifts.
Microscopic Model

- Each term of the Schwinger model can be implemented via a Bose-Fermi Hubbard model and using superlattices.
- What is a superlattice? – optical potential created by superposition of different harmonics.

- Fermion mass can be directly implemented with super-lattice.
- For link fields need Schwinger boson(s).
Microscopic Model

Label bosonic d.o.f on even and odd links as \( a \) and \( b \)

\[
\begin{align*}
U_{2x,2x+1} &= S^+_{2x,2x+1} = a^+_{2x} a_{2x+1} \\
U_{2x-1,2x} &= S^+_{2x-1,2x} = b^+_{2x-1} b_{2x} \\
E_{2x,2x+1} &= S^z_{2x,2x+1} = (n^a_{2x} - n^a_{2x+1})/2 \\
E_{2x-1,2x} &= S^z_{2x-1,2x} = (n^b_{2x-1} - n^b_{2x})/2 \\
n^a_{2x} + n^a_{2x+1} &= 2S = n^b_{2x-1} + n^b_{2x}
\end{align*}
\]

For a spin-1 representation, for example,
Consider the following interaction term for the Hubbard Hamiltonian:

\[ H_U = U \sum_x \left[ (n^a_x)^2 + (n^b_x)^2 + 2n_x(n_a + n_b) + 2n_a n_b - (-1)^x (n_x + n^a_x + n^b_x) \right] \]

\[ = 2U \sum_x \left[ (S^z_{x,x+1})^2 + (n_x - n_{x+1}) S^z_{x,x+1} - S^z_{x-1,x} S^z_{x,x+1} - (-1)^x (S^z_{x,x+1} + n_x/2) \right] \]

\[ = U \sum_x (G_x)^2 \]

where we used

\[ 4(S^z_{2x,2x+1})^2 = n^a_{2x} + n^a_{2x-1} - 2n^a_{2x} n^a_{2x-1} \]

\[ 4(S^z_{2x-1,2x})^2 = n^b_{2x} + n^b_{2x-1} - 2n^b_{2x} n^b_{2x-1} \]

\[ (n^a_{2x})^2 + (n^a_{2x+1})^2 = 4S^2 - 2n^a_{2x} n^a_{2x+1} \]

and ignored constants.

Violating GI costs energy \( \mathcal{O}(U) \)

In the limit \( U \to \infty \) GI exact!
Low energy physics

Low energy physics induced by $H_{pert}$:

$$H_{pert} = t_F \sum_x (\psi_x^\dagger \psi_{x+1} + h.c) + m \sum_x (-1)^x n_x + \frac{g^2}{4} \sum_x \left[(n_a^x)^2 + (n_b^x)^2\right]$$

$$+ \frac{J}{2} \sum_{x \in odd} (b_x^\dagger b_{x+1} + h.c) + \frac{J}{2} \sum_{x \in even} (a_x^\dagger a_{x+1} + h.c)$$

Other possible GI states are also generated: in particular, the fermion-gauge coupling is generated in 2nd order PT

$$\sim -\frac{t_F J}{2U}$$

$$\psi_{2x}^\dagger S_{2x-1,2x}^- \psi_{2x-1}$$

The other contribution suppressed by $\frac{t_F}{J}$
How good is the approximation?

Prob of remaining in the GI subspace better than 98% for $\frac{J}{U} = \frac{1}{20}$!

Analysis of string breaking in the effective model for spin-1 under progress...
The SO(3) QLM

• The Hamiltonian of the model is

\[ H = -J \sum_{x, \mu \neq \nu} \text{Tr}(U_{x, \mu} U_{x+\hat{\mu}, \nu} U_{x+\hat{\nu}, \mu}^\dagger U_{x, \mu}^\dagger + \text{h.c.}) \]

• In the SO(3) representation, \( U_{x, \mu} \) are 3 \times 3 matrices.

\( (U_{i,j}, \ i = 1, 2, 3; \ j = 1, 2, 3) \)

• Left (\( \tilde{L} \)) and the right (\( \tilde{R} \)) generators of GT distinct, satisfy the following commutation relations:

\[ [R^a, U_{ij}] = -2i\epsilon_{akj} U_{ik}; \quad [L^a, U_{ij}] = 2i\epsilon_{aik} U_{kj} \]

An elegant representation can be obtained for all operators in terms of the \( \sigma \)-matrices

\[ L^a = \sigma^a_L; \quad R^a = \sigma^a_R; \quad U_{ij} = \sigma^i_L \sigma^j_R \]

Four states per link
The SO(3) QLM: Spectrum

- Non-abelian Gauss’ law: \( \vec{G}_x = \sum_\mu (\vec{R}_x - \hat{x}_\mu + \vec{L}_x,\mu) \) requires construction of Gauge singlets
- Construct singlets out of 2d spin-1/2

\[
\left( \begin{array}{c} 1 \\ 2 \\ \frac{1}{2} \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 2 \\ \frac{1}{2} \end{array} \right) = \left( \begin{array}{c} 1 \\ 2 \\ \frac{1}{2} \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 2 \\ \frac{1}{2} \end{array} \right) = \left( \begin{array}{c} 0 \oplus 1 \\ 0 \oplus 1 \oplus 1 \oplus 0 \oplus 1 \oplus 2 \end{array} \right)
\]

2 gauge inv states/site: effective spin-1/2 system (again!) \( \Rightarrow \) Total no of states = \( 2L^2 \)
The SO(3) QLM: Screening

Triplet (spin-1) charges

V(x) vs. x

-8.0
-7.5
-7.0
-6.5
-6.0
-5.5
-5.0
-4.5
-4.0
-3.5
-3.0
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
1.0
2.0
3.0
4.0
5.0

8x2
10x2
The SO(3) QLM: Screening

Quintet (spin-2) charges

V(x)

x

V(x) vs. x for different representations: 8x2 (squares) and 10x2 (circles).
Outlook

• Disclaimer: Far from the continuum limit.
• Extension to higher dimensions in principle straightforward (note: no bosonic representation used for the fermions)
• At the starting point, even qualitative results useful
• Validation of quantum simulations will need MC simulations to check them
• Development of new algorithms ... (cluster algorithms)
• More sophisticated models ... formulation of full QCD in terms of QLMs already exist  Brower Chandrasekharan Wiese (1997)