HADRONIC LIGHT-BY-LIGHT: 
EXTENDED NAMBU-JONA-LASINIO 
AND CHIRAL QUARK MODELS

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Various ChPT: http://www.theplu.se/~bijnens/chpt.html
Overview

- Ximo and our $g-2$ related work
- What do we calculate and some general statements
- ENJL
- A note on MV short-distance and the quark loop
- Quark-loop: irreducible four point-functions
- Scalar exchange
- Pseudo-scalar exchange
- Axial vector exchange
- $\pi$ and $K$ loop
- Summary
- More recent and some comments
In memoriam: Joaquim Prades

Dedicated to

Ximo Prades 1963-2010

Friend and collaborator

Postdoc 93-95
with me in Copenhagen
we have worked together ever since
We have worked together on $g - 2$, $\Delta I = 1/2$, $B_K$, $\varepsilon'/\varepsilon$, Quark models and ENJL, electromagnetic effects, . . . and were working on rare kaon decays and $g - 2$.


Our object

Muon line and photons: well known
The blob: fill in with hadrons/QCD
Trouble: low and high energy very mixed
Double counting needs to be avoided: hadron exchanges versus quarks
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
- $p^4$, order 1: pion-loop
- $p^8$, order $N_c$: quark-loop and heavier exchanges
- $p^6$, order $N_c$: pion exchange

Does not fully solve the problem
only short-distance quark-loop is really $p^8$
but it’s a start
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
- $p^4$, order 1: pion-loop
- $p^8$, order $N_c$: quark-loop and heavier exchanges
- $p^6$, order $N_c$: pion exchange

- Hayakawa, Kinoshita, Sanda: meson models, pion loop using hidden local symmetry, quark-loop with VMD, calculation in Minkowski space
- JB, Pallante, Prades: This talk but try using as much as possible a consistent model-approach, calculation in Euclidean space
Papers: BPP and HKS

JB, E. Pallante and J. Prades


Hayakawa, Kinoshita, (Sanda)


Differences

- HK(S)
  - Purely hadronic exchanges
  - quark-loop with hadronic VMD
  - Studied dependence on everything on $m_V$

- BPP
  - Use the ENJL as an overall model to have a similar uncertainty on all low-energy parts
  - repair some of the worst short-comings
  - Add the short-distance quark-loop
  - Large study of cut-off dependence
Differences

HK(S)
- Purely hadronic exchanges
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Sign mistake
- HKS: Euclidean versus Minkowski $\varepsilon^{\mu\nu\alpha\beta}$
- BPP: notes all correct sign, program had wrong sign, probably minus sign from fermion loop not removed
The overall

\[ a_\mu^{\text{light-by-light}} = \frac{1}{48 m_\mu} \text{tr}[(\not p + m_\mu) M^{\lambda\beta}(0) (\not p + m_\mu) [\gamma_\lambda, \gamma_\beta]]. \]

\[ M^{\lambda\beta}(p_3) = |e|^6 \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \frac{1}{q^2 p_1^2 p_2^2 (p_4^2 - m_\mu^2)(p_5^2 - m_\mu^2)} \]
\[ \times \left[ \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_3\lambda} \right] \gamma_\alpha (\not p_4 + m_\mu) \gamma_\nu (\not p_5 + m_\mu) \gamma_\rho. \]

We used: \( \Pi^{\rho\nu\alpha\lambda}(p_1, p_2, p_3) = -p_3\beta \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_3\lambda}. \)

Can calculate at \( p_3 = 0 \) but must take derivative

derivative makes in quark-loop each permutation finite

Four point function of \( V_i^\mu(x) \equiv \sum_i Q_i \left[ \bar{q}_i(x) \gamma^\mu q_i(x) \right] \)

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \equiv \]
\[ i^3 \int d^4 x \int d^4 y \int d^4 z e^{i(p_1 \cdot x + p_2 \cdot y + p_3 \cdot z)} \langle 0 | T \left( V_\rho^\alpha(0) V_\nu^\alpha(x) V_\beta^\alpha(y) V_\rho^\beta(z) \right) | 0 \rangle \]
General properties

$\Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)$:

- In general 138 Lorentz structures (but only 32 contribute to $g - 2$)

- Using $q_\rho \Pi^{\rho\nu\alpha\beta} = p_{1\nu} \Pi^{\rho\nu\alpha\beta} = p_{2\alpha} \Pi^{\rho\nu\alpha\beta} = p_{3\beta} \Pi^{\rho\nu\alpha\beta} = 0$
- 43 gauge invariant structures

- Bose symmetry relates some of them

\[ \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \]

- 8 dimensional integral, three trivial,

- 5 remain: $p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu$

- Rotate to Euclidean space:
  - Easier separation of long and short-distance
  - Artefacts (confinement) in models smeared out.
ENJL: our main model

\[ L_{\text{ENJL}} = \bar{q}^\alpha \left\{ i \gamma^\mu \left( \partial_\mu - i v_\mu - ia_\mu \gamma_5 \right) - (\mathcal{M} + s - ip\gamma_5) \right\} q^\alpha + 2 g_S \left( \bar{q}^\alpha_R q^\beta_L \right) \left( \bar{q}^\beta_L q^\alpha_R \right) \]

\[ -g_V \left[ \left( \bar{q}^\alpha_L \gamma^\mu q^\beta_L \right) \left( \bar{q}^\beta_L \gamma_\mu q^\alpha_L \right) + \left( \bar{q}^\alpha_R \gamma^\mu q^\beta_R \right) \left( \bar{q}^\beta_R \gamma_\mu q^\alpha_R \right) \right] \]

- \( \bar{q} \equiv (\bar{u}, \bar{d}, \bar{s}) \)
- \( v_\mu, a_\mu, s, p \): external vector, axial-vector, scalar and pseudoscalar matrix sources
- \( \mathcal{M} \) is the quark-mass matrix.
- \( g_V \equiv \frac{8\pi^2 G_V(\Lambda)}{N_c \Lambda^2} \), \( g_S \equiv \frac{4\pi^2 G_S(\Lambda)}{N_c \Lambda^2} \).
- \( G_V, G_S \) are dimensionless and valid up to \( \Lambda \)
- No confinement but has good pion, vector meson and OK axial vector-meson phenomenology
ENJL: our main model


- Gap equation: chiral symmetry spontaneously broken

\[
\begin{align*}
\text{--} & \quad = \quad \text{--} + \quad \circ
\end{align*}
\]

- Generates poles, i.e. mesons via bubble resummation
ENJL: our main model

- Can be thought of as a very simple rainbow and ladder approximation in the DSE equation with constant kernels for the one-gluon exchange
- Parameters fit via $F_\pi$, $L_i$, vector meson properties, . . .
- $G_S = 1.216$, $G_V = 1.263$, $\Lambda = 1.16$ GeV
- has $M_Q = 263$ MeV
- Has a number of decent matchings to short-distance, e.g. $\Pi_V - \Pi_A$ but fails in others.
- Generates always VMD in external legs (but with a twist)
- Hook together general processes by one-loop vertices and bubble-chain propagators
Separation of contributions

- Quark loop with external bubble-chains
- $\approx$ Quark-loop with VMD

- Also internal bubble chain
- $\approx$ meson exchange
- Note that vertices have structure
- Off-shell effect in model included
MV short-distance and quark-loop


- Take $p_1^2 \approx p_2^2 \gg q^2$: Leading term in OPE of two vector currents is proportional to axial current

- These come from

- Are these part of the quark-loop? See also in Dorokhov, Broniowski, phys. Rev. D78(2008)07301


- Ximo noticed: large $N_c$ MV $\approx$ large $N_c$ BPP
## Pure quark loop

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^7$ Electron Loop</th>
<th>$a_\mu \times 10^9$ Muon Loop</th>
<th>$a_\mu \times 10^9$ Constituent Quark Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.41(8)</td>
<td>2.41(3)</td>
<td>0.395(4)</td>
</tr>
<tr>
<td>0.7</td>
<td>2.60(10)</td>
<td>3.09(7)</td>
<td>0.705(9)</td>
</tr>
<tr>
<td>1.0</td>
<td>2.59(7)</td>
<td>3.76(9)</td>
<td>1.10(2)</td>
</tr>
<tr>
<td>2.0</td>
<td>2.60(6)</td>
<td>4.54(9)</td>
<td>1.81(5)</td>
</tr>
<tr>
<td>4.0</td>
<td>2.75(9)</td>
<td>4.60(11)</td>
<td>2.27(7)</td>
</tr>
<tr>
<td>8.0</td>
<td>2.57(6)</td>
<td>4.84(13)</td>
<td>2.58(7)</td>
</tr>
<tr>
<td>Known Results</td>
<td>2.6252(4)</td>
<td>4.65</td>
<td>2.37(16)</td>
</tr>
</tbody>
</table>

$M_Q : 300$ MeV

now all known analytically

Us: $5+(3-1)$ integrals

extra are Feynman parameters

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**Slow convergence:**

- electron: all at 500 MeV
- Muon: only half at 500 MeV, at 1 GeV still 20% missing
- 300 MeV quark: at 2 GeV still 25% missing
\[ \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3) = \Pi(p_1, p_2, p_3) \nu^{\rho \nu \alpha \beta}(p_1, p_2, p_3) \]

\[ \nu^{abcd \rho \nu \alpha \beta}(p_1, p_2, p_3) = \left( \frac{g^{a \rho} M_V^2 (-q^2) - q^a q^\rho}{M_V^2 (-q^2) - q^2} \right) \left( \frac{g^{b \nu} M_V^2 (-p_1^2) - p_1^b p_1^\nu}{M_V^2 (-p_1^2) - p_1^2} \right) \]

\[ \times \left( \frac{g^{c \alpha} M_V^2 (-p_2^2) - p_2^c p_2^\alpha}{M_V^2 (-p_2^2) - p_2^2} \right) \left( \frac{g^{d \beta} M_V^2 (-p_3^2) - p_3^d p_3^\beta}{M_V^2 (-p_3^2) - p_3^2} \right) \]

- Barred = one-loop but need to add all permutations
- the extra terms with \( q^\alpha q^\rho, \ldots \) vanish because of one-loop gauge invariance
- remainder amounts to \( \times \frac{M_V(q^2)}{M_V(q^2) - q^2} \) on each leg
## ENJL quark-loop

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ Quark-loop VMD</th>
<th>$a_\mu \times 10^{10}$ Quark-loop ENJL</th>
<th>$a_\mu \times 10^{10}$ Quark-loop masscut</th>
<th>$a_\mu \times 10^{10}$ ENL+masscut sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>2.46</td>
<td>3.2</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>1.13</td>
<td>2.3</td>
</tr>
<tr>
<td>1.0</td>
<td>0.87</td>
<td>1.44</td>
<td>0.59</td>
<td>2.0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
<td>0.13</td>
<td>1.9</td>
</tr>
<tr>
<td>4.0</td>
<td>0.98</td>
<td>1.98</td>
<td>0.03</td>
<td>2.0</td>
</tr>
<tr>
<td>8.0</td>
<td>0.98</td>
<td>2.00</td>
<td>0.005</td>
<td>2.0</td>
</tr>
</tbody>
</table>

- ENJL cuts off slower than pure VMD
- **masscut:** $M_Q = \Lambda$ to have short-distance and no problem with momentum regions
- Quite stable in region 1-4 GeV

**Very stable**
ENJL: scalar

\[ \Pi^{\mu \nu \alpha \beta}(p_1, p_2, p_3) = \Pi_{ab}^{VVV}(p_1, r) g_S \left( 1 + g_S \Pi^S(r) \right) \Pi_{cd}^{SVV}(p_2, p_3) \nu^{abcd\rho \nu \alpha \beta}(p_1, p_2, p_3) \]

+ permutations

\[ g_S \left( 1 + g_S \Pi_S \right) = \frac{g_A(q^2)(2M_Q)^2}{2f^2(q^2)} \frac{1}{M_S^2(q^2) - q^2} \]

\nu^{abcd\rho \nu \alpha \beta} \text{ was ENJL VMD legs}

In ENJL only scalar+quark-loop properly chiral invariant
**ENJL: scalar**

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ Quark-loop VMD</th>
<th>$a_\mu \times 10^{10}$ Quark-loop ENJL</th>
<th>$a_\mu \times 10^{10}$ Scalar Exchange</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>$-0.22$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>$-0.46$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.87</td>
<td>1.44</td>
<td>$-0.60$</td>
</tr>
<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
<td>$-0.68$</td>
</tr>
<tr>
<td>4.0</td>
<td>0.98</td>
<td>1.98</td>
<td>$-0.68$</td>
</tr>
<tr>
<td>8.0</td>
<td>0.98</td>
<td>2.00</td>
<td>$-0.68$</td>
</tr>
</tbody>
</table>

- Note: ENJL+scalar similar to Quark-loop VMD
- $M_S \approx 620$ MeV so certainly an overestimate for real scalars
- Is why ENJL has $\pi\pi$-scattering very well
- If scalar is $\sigma$: related to pion loop part?
ENJL: pseudo-scalar

- ENJL needs added pieces to get anomaly correct
- Many quark models with cut-off need this: JB, Prades, 1993
- Pure ENJL does not reproduce CLEO $P\gamma\gamma^*$ data
  - Tamper with the ENJL VMD factor (transverse VMD)
- Pure ENJL is large $N_c$: 9 Goldstone bosons:
  3 neutral: $\pi^0, \pi_1 \sim \bar{u}u + \bar{d}d, \pi_2 \sim \bar{s}s$.
  - Our solutions: take ENJL for $\pi^0$ and keep ratio $\pi^0, \eta$ and $\eta'$ using double VMD+pointlike propagator
  - All large $N_c$ models face this problem
The $\pi^0\gamma^*\gamma$ form factor. At high energies the curves from top to bottom are: ENJL, ENJL-VMD, Point-like-VMD and Transverse-VMD. Pointlike is a straight line at the top.
ENJL: pseudo-scalar

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) = \left[ \prod^{VV\mu}(p_1, r) \left( 1 + g_S \Pi^P(r) \right) \prod^{PV\mu}(p_2, p_3) \right. \\
- g_V \prod^{VV\mu}(p_1, r) \prod^{AV\mu}(p_2, p_3) - g_V \prod^{VV\mu}(p_1, r) \prod^{PV\mu}(-r) \prod^{PV\mu}(p_2, p_3) \right] \\
\times g_S \nu^{\rho\nu\alpha\beta}(p_1, p_2, p_3) + \cdots \\

BLUE: extra anomaly pieces

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) = \\
g_S \prod^{VV\mu}(p_1, r) \left( 1 + g_S \Pi^P(r) - 4g_V M_i \Pi^P_M(r^2) \right) \prod^{PV\mu}(p_2, p_3) \nu^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \\
+ 2M_Q g_S g_V \Pi^P_M(r^2) \left[ \prod^{VV\mu}(p_1, r) \left\{ \prod^{PV\mu}(p_2, p_3) \big|_{p_2^2=p_3^2=r^2=0} \right\} \\
\times \left( \frac{g^{\alpha\rho} M_V^2(-q^2) - q^\alpha q^\rho}{M_V^2(-q^2) - q^2} \right) \left( \frac{g^{\nu\beta} M_V^2(-p_1^2) - p_1^\nu p_1^\beta}{M_V^2(-p_1^2) - p_1^2} \right) g^{\alpha\beta} \right] \\
+ \left\{ \prod^{VV\mu}(p_1, r) \big|_{p_1^2=r^2=q^2=0} \right\} \prod^{PV\mu}(p_2, p_3) \\
\times g^{\alpha\rho} g^{\nu\beta} \left( \frac{g^{\alpha\beta} M_V^2(-p_2^2) - p_2^\alpha p_2^\beta}{M_V^2(-p_2^2) - p_2^2} \right) \left( \frac{g^{\nu\beta} M_V^2(-p_3^2) - p_3^\nu p_3^\beta}{M_V^2(-p_3^2) - p_3^2} \right) \right] + \cdots , \]
\( \pi^0 \) exchange contributions with various parametrizations of the \( \pi^0 \gamma^* \gamma^* \) vertex that fit the data for the \( \pi^0 \gamma \gamma^* \) vertex. All in reasonable agreement.

<table>
<thead>
<tr>
<th>Cut-off ( \mu ) (GeV)</th>
<th>( a_\mu \times 10^{10} ) Point-like</th>
<th>( a_\mu \times 10^{10} ) ENJL–VMD</th>
<th>( a_\mu \times 10^{10} ) Point-Like- VMD</th>
<th>( a_\mu \times 10^{10} ) Transverse- VMD</th>
<th>( a_\mu \times 10^{10} ) Transverse- VMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4.92(2)</td>
<td>3.29(2)</td>
<td>3.46(2)</td>
<td>3.60(3)</td>
<td>3.53(2)</td>
</tr>
<tr>
<td>0.7</td>
<td>7.68(4)</td>
<td>4.24(4)</td>
<td>4.49(3)</td>
<td>4.73(4)</td>
<td>4.57(4)</td>
</tr>
<tr>
<td>1.0</td>
<td>11.15(7)</td>
<td>4.90(5)</td>
<td>5.18(3)</td>
<td>5.61(6)</td>
<td>5.29(5)</td>
</tr>
<tr>
<td>2.0</td>
<td>21.3(2)</td>
<td>5.63(8)</td>
<td>5.62(5)</td>
<td>6.39(9)</td>
<td>5.89(8)</td>
</tr>
<tr>
<td>4.0</td>
<td>32.7(5)</td>
<td>6.22(17)</td>
<td>5.58(5)</td>
<td>6.59(16)</td>
<td>6.02(10)</td>
</tr>
</tbody>
</table>
Pseudoscalar exchange

<table>
<thead>
<tr>
<th>Cut-off $\mu$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ ENJL</th>
<th>$a_\mu \times 10^{10}$ Point-Like–VMD ($\pi^0$)</th>
<th>$a_\mu \times 10^{10}$ $\eta$</th>
<th>$a_\mu \times 10^{10}$ $\eta'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>2.84(2)</td>
<td>2.70(1)</td>
<td>0.425(1)</td>
<td>0.266(1)</td>
</tr>
<tr>
<td>0.5</td>
<td>3.70(3)</td>
<td>3.46(2)</td>
<td>0.616(2)</td>
<td>0.399(2)</td>
</tr>
<tr>
<td>0.7</td>
<td>5.04(4)</td>
<td>4.49(3)</td>
<td>0.923(3)</td>
<td>0.631(2)</td>
</tr>
<tr>
<td>1.0</td>
<td>6.44(7)</td>
<td>5.18(3)</td>
<td>1.180(4)</td>
<td>0.847(3)</td>
</tr>
<tr>
<td>2.0</td>
<td>8.83(17)</td>
<td>5.62(5)</td>
<td>1.37(1)</td>
<td>1.03(1)</td>
</tr>
<tr>
<td>4.0</td>
<td>10.51(37)</td>
<td>5.58(5)</td>
<td>1.38(1)</td>
<td>1.04(1)</td>
</tr>
</tbody>
</table>

The pseudoscalar exchange contribution to $a_\mu$ for the ENJL (large $N_c$) and the point-like Wess-Zumino vertex, damped with two vector propagators for the $\pi^0$, $\eta$ and $\eta'$. 

Axial-vector exchange exchange

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ from Axial-Vector Exchange $\mathcal{O}(N_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.05(0.01)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.07(0.01)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.13(0.01)</td>
</tr>
<tr>
<td>2.0</td>
<td>0.24(0.02)</td>
</tr>
<tr>
<td>4.0</td>
<td>0.59(0.07)</td>
</tr>
</tbody>
</table>

There is some pseudo-scalar exchange piece here as well, off-shell not quite clear what is what.
\( \pi \) and \( K \)-loop

- The \( \pi\pi\gamma^* \) vertex is always done using VMD
- \( \pi\pi\gamma^*\gamma^* \) vertex two choices:
  - Hidden local symmetry model (only one \( \gamma \) has VMD
  - Full VMD
  - Both are chirally symmetric
  - Check if they live up to MV short distance (Full VMD does, HLS not checked yet)
- The HLS model used has problems with \( \pi^+-\pi^0 \) mass difference (due not having an \( a_1 \))
- Final numbers quite different: \(-0.045\) and \(-0.19\)
- For us stopped at 1 GeV but within 10\% of higher \( \Lambda \)
\( \pi \) and \( K \)-loop

<table>
<thead>
<tr>
<th>Cut-off GeV</th>
<th>( 10^{10} a_\mu ) ( \pi ) bare</th>
<th>( 10^{10} a_\mu ) ( \pi ) VMD</th>
<th>( 10^{10} a_\mu ) ( \pi ) ENJL</th>
<th>( 10^{10} a_\mu ) ( K ) ENJL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-1.71(7)</td>
<td>-1.16(3)</td>
<td>-1.20(0.03)</td>
<td>-0.020(0.001)</td>
</tr>
<tr>
<td>0.6</td>
<td>-2.03(8)</td>
<td>-1.41(4)</td>
<td>-1.42(0.03)</td>
<td>-0.026(0.001)</td>
</tr>
<tr>
<td>0.7</td>
<td>-2.41(9)</td>
<td>-1.46(4)</td>
<td>-1.56(0.03)</td>
<td>-0.034(0.001)</td>
</tr>
<tr>
<td>0.8</td>
<td>-2.64(9)</td>
<td>-1.57(6)</td>
<td>-1.67(0.04)</td>
<td>-0.042(0.001)</td>
</tr>
<tr>
<td>1.0</td>
<td>-2.97(12)</td>
<td>-1.59(15)</td>
<td>-1.81(0.05)</td>
<td>-0.048(0.002)</td>
</tr>
<tr>
<td>2.0</td>
<td>-3.82(18)</td>
<td>-1.70(7)</td>
<td>-2.16(0.06)</td>
<td>-0.087(0.005)</td>
</tr>
<tr>
<td>4.0</td>
<td>-4.12(18)</td>
<td>-1.66(6)</td>
<td>-2.18(0.07)</td>
<td>-0.099(0.005)</td>
</tr>
</tbody>
</table>

- We ran HLS but those data I don’t find anymore
- note the suppression by the propagators
\[ a_\mu^{\text{LbL}} = (2.1 \pm 0.3) \cdot 10^{-10} \text{[quark-loop]} \\
+ (-0.68 \pm 0.2) \cdot 10^{-10} \text{[scalar]} \\
+ (8.5 \pm 1.3) \cdot 10^{-10} \text{[pseudoscalar]} \\
+ (0.25 \pm 0.1) \cdot 10^{-10} \text{[axial-vector]} \\
+ (-1.9 \pm 1.3) \cdot 10^{-10} \text{[}\pi K\text{-loop]} \\
= (8.3 \pm 3.2) \cdot 10^{-10}. \]
Since then

Constraints from experiment: J. Bijnens and F. Persson, hep-ph/hep-ph/0106130 Studying three formfactors $P\gamma^*\gamma^*$ in $P \rightarrow \ell^+\ell^-\ell'^+\ell'^-$, $e^+e^- \rightarrow e^+e^-P$ exact tree level and for $g - 2$ (but beware sign):

- Conclusion: possible but VERY difficult
- Two $\gamma^*$ off-shell not so important for our choice of form-factor

Sign mistake found and a more analytical evaluation of $\pi^0$ exchange M. Knecht, A. Nyffeler, Hadronic light by light corrections to the muon $g-2$: The Pion pole contribution, Phys. Rev. D65(2002)073034, [hep-ph/0111058]

Since then

- More short-distance constraints: see later talks by Vainshtein and Nyffeler

- Chiral nonlocal quark-model (like nonlocal ENJL):

\[
\mathcal{L} = \bar{q}^\alpha(x) \left\{ i \gamma^\mu \left( \partial_\mu - i v_\mu - i a_\mu \gamma_5 \right) - (\mathcal{M} + s - i p \gamma_5) \right\} q^\alpha(x)
\]

\[
+ \frac{1}{2} G_P \left( \Pi_{i=1,4} \int d^4 x_i f(x_i) \right) \left( \bar{q}^\alpha(x - x_1) \Gamma q^\alpha(x + x_3) \right) \left( \bar{q}^\beta(x - x_2) \Gamma q^\beta(x + x_4) \right)
\]

- \( \Gamma \otimes \Gamma = 1 \otimes 1 - \gamma_5 \tau^a \otimes \gamma_5 \tau^a \) and \( f(p) = e^{-p^2/\Lambda^2} \)

- Path ordered external fields part not written

- \( a_{\mu}^{\pi_0} = 6.27 \times 10^{-10} \), very much like \( \pi^0 \) exchange with double VMD
Since then

- The ENJL model can certainly be improved: Nonlocal model, so far only $\pi^0$-exchange DSE approach, first results see talk by Williams, published only $\pi^0$ exchange. Both only $SU(2)$ as well so far.

- Variations on hadronic models: AdS/QCD: talk by Cata

- Ximo and I were working on getting a more consistent matching between the different regimes
Since then

For that we need to look at more figures like:

\[ a_\mu = \int dl_1 dl_2 a^{LL}_\mu \text{ with } l_i = \log\left(\frac{P_i}{GeV}\right) \]

Checking which momentum regions do what (but would need three dimensional)
Need a new overall evaluation with consistent approach.

More resonances models should be tried, AdS/QCD is one approach, $R_X T$ (Valencia et al.) possible,…

But note that short-distance matching here must be done in many channels and there are theorems JB,Gamiz,Lipartia,Prades that with only a few resonances this requires compromises

Pion loop needs a bit more study, why is HLS smaller than double VMD. HLS model with $\rho$ and $a_1$ ?

More general theorems on the signs of contributions like scalar negative, pseudo-scalar positive?

Cancellations between higher mass chiral partners?

Can (hard pion) ChPT help (also for lattice)?