QCD sum rules concepts and pion’s structure

Alexander Pimikov
in collaboration with A. Bakulev, S. Mikhailov, and N. Stefanis

Bogoliubov Lab. Theor. Phys., JINR (Dubna, Russia)
ITP-II, Ruhr-Universität (Bochum, Germany)
Outline:

- QCD sum rules (SR) approach
- Introducing Nonlocal Condensates in OPE
- Pion DA, moments, end-point behavior of pion DA
- Electromagnetic pion FF in space-like region
- Pion-photon transition FF in LC SR
- Pion DA from experiment
- Conclusions
QCD SR Approach

Determination of spectrum parameters from requirement of agreement between two ways for correlator $\Pi(Q^2)$ of currents:

1. 1th way — Dispersion relation: decay constants $f_h$, masses $m_h$ and others,

$$\Pi_{\text{had}}(Q^2) = \int_0^\infty \frac{\rho_{\text{had}}(s)}{s + Q^2} \, ds + \text{subtractions}.$$

2. model spectral density:

$$\rho_{\text{had}}(s) = f_h^2 \delta(s - m_h^2) + \rho_{\text{pert}}(s) \theta(s - s_0).$$

Frontiers in QCD, INT, 2011
Theoretical part of QCD SR

2th way — Operator product expansion:

\[ \Pi_{OPE}(Q^2) = \Pi_{pert}(Q^2) + \sum_n C_n \frac{\langle 0| : O_n : |0 \rangle}{Q^{2n}}. \]

Condensates \( \langle 0| : O_n : |0 \rangle \equiv \langle O_n \rangle = ? \) (next slides).
QCD SR Approach

Determination of spectrum parameters from requirement of agreement between two ways for correlator:

1th way — Dispersion relation: decay constants $f_h$ and masses $m_h$,

$$\Pi_{\text{had}}(Q^2) = \int_0^\infty \frac{\rho_{\text{had}}(s) ds}{s + Q^2} + \text{subtractions}.$$  

model spectral density: $\rho_{\text{had}}(s) = f_h^2 \delta (s - m_h^2) + \rho_{\text{pert}}(s) \theta (s - s_0)$.

2th way — Operator product expansion:

$$\Pi_{\text{OPE}}(Q^2) = \Pi_{\text{pert}}(Q^2) + \sum_n C_n \frac{\langle 0 | : O_n : | 0 \rangle}{Q^{2n}}.$$  

Condensates $\langle 0 | : O_n : | 0 \rangle \equiv \langle O_n \rangle = ?$ (next slides).

QCD SR reads:

$$\Pi_{\text{had}}(Q^2, m_h, f_h) = \Pi_{\text{OPE}}(Q^2).$$
Borel Transform

\[ \Phi(M^2) = \hat{B}_{Q^2 \to M^2} \left[ \Pi(Q^2) \right] = \lim_{n \to \infty} \frac{(-Q^2)^n}{\Gamma(n)} \left[ \frac{d^n}{dQ^{2n}} \Pi(Q^2) \right]_{Q^2 = nM^2}. \]

<table>
<thead>
<tr>
<th>( \Pi(Q^2) )</th>
<th>( C = \text{const} )</th>
<th>( Q^{2n} )</th>
<th>( 1/Q^{2n} )</th>
<th>( 1/ \left( s + Q^2 \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi(M^2) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( e^{-s/M^2} / M^2 )</td>
</tr>
</tbody>
</table>

- Elimination of subtractions in dispersion relation
- Exponential suppression of higher states contribution
- Factorial suppression of condensate terms

\[ \int_0^2 f_h^2 e^{-m_h^2/M^2} + \int_{s_0}^{\infty} \rho_{\text{pert}}(s) e^{-s/M^2} ds \]

\[ = \int_0^{\infty} \rho_{\text{pert}}(s) e^{-s/M^2} ds + \frac{c_G}{M^2} \frac{\alpha_s}{\pi} G_{\mu\nu}^a G^{a\mu\nu} + \frac{c_{\bar{q}q}}{M^4} \alpha_s \langle \bar{q}q \rangle^2. \]
Borel Transform

\[ \Phi(M^2) = \hat{B}_{Q^2 \to M^2} \left[ \Pi(Q^2) \right] = \lim_{n \to \infty} \frac{(-Q^2)^n}{\Gamma(n)} \left[ \frac{d^n}{dQ^{2n}} \Pi(Q^2) \right]_{Q^2 = nM^2}. \]

<table>
<thead>
<tr>
<th>(\Pi(Q^2))</th>
<th>(C = \text{const})</th>
<th>(Q^{2n})</th>
<th>(1/Q^{2n})</th>
<th>(1/\left(s + Q^2\right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi(M^2))</td>
<td>0</td>
<td>0</td>
<td>1/(\left(\Gamma(n)M^{2n}\right))</td>
<td>(e^{-s/M^2}/M^2)</td>
</tr>
</tbody>
</table>

- Elimination of subtractions in dispersion relation
- Exponential suppression of higher states contribution
- Factorial suppression of condensate terms

\[ f_h^2 e^{-m_n^2/M^2} = \int_{0}^{s_0} \rho_{\text{pert}}(s) e^{-s/M^2} ds + \frac{c_G}{M^2} \frac{\alpha_s}{\pi} \langle G^{\alpha \nu} G^{\alpha \mu \nu} \rangle + \frac{cqq}{M^4} \alpha_s \langle \bar{q}q \rangle^2. \]
Introducing condensates in QCD calculations

\[
\langle 0 | T (\bar{q}_B(0)q_A(x)) | 0 \rangle = \langle 0 | :\bar{q}_B(0)q_A(x): | 0 \rangle - i\hat{S}_{AB}(x)
\]

QCD PT

\[\langle \bar{q}q \rangle \overset{\text{def}}{=} 0\]

QCD SR

\[\langle \bar{q}_A(0)q_A(x) \rangle = \langle \bar{q}q \rangle\]

\[\text{CONST} \neq 0\]

\[\text{SVZ’79}\]

Condensate

Decay constants, masses of hadrons

NLC QCD SR

\[\langle \bar{q}(0)q(x) \rangle = F_S(x^2) + \hat{x}F_V(x^2)\]

\[\text{M&R ’86}\]

Nonlocal condensate

Distribution Amplitudes, Form Factors

\[\langle \bar{q}_B(0)q_A(x) \rangle = \delta_{AB} \frac{x^2}{4} \left[ \langle \bar{q}q \rangle + \frac{x^2}{4} \frac{\langle \bar{q}D^2q \rangle}{2} + \ldots \right] + i\hat{x}_{AB} \frac{x^2}{4} \left[ \frac{2\alpha_s \pi \langle \bar{q}q \rangle^2}{81} + \ldots \right].\]
Diagrams for $\langle T (J_{\nu}(z) J_{\mu}(0)) \rangle$

Quarks run through vacuum with nonzero momentum $k \neq 0$:

$$2\langle k^2 \rangle = \frac{\langle \bar{q}D^2q \rangle}{\langle \bar{q}q \rangle} = \lambda_q^2 = 0.40(5) \text{ GeV}^2$$
Coordinate dependence of condensates

Parameterization for scalar condensate was suggested in works of Bakulev, Mikhailov and Radyushkin:

\[
\langle : \bar{q}_A(0)q_A(x) : \rangle = \langle \bar{q}q \rangle \int_0^\infty f_S(\alpha) e^{\alpha x^2/4} d\alpha , \text{ where } x^2 < 0.
\]

First approximation which takes into account finite width of quark distribution in vacuum: \( f_S(\alpha) = \delta (\alpha - \lambda_q^2/2) , \quad \lambda_q^2 = \langle \bar{q}D^2q \rangle / \langle \bar{q}q \rangle \).

Such representation corresponds to Gaussian form \( \sim \exp (\lambda_q^2 x^2 / 8) \) of NLC in coordinate representation.

The heavy-quark effective theory (Radyushkin 91) tells us that the scalar condensate decreases exponentially at large distances.

The smooth model \( f_S(\alpha) \sim \alpha^{n-1} \exp (-\Lambda^2/\alpha - \sigma^2 \alpha) \) has a sensible asymptotic form \( \langle \bar{q}(0)q(x) \rangle \bigg|_{x^2 \to \infty} \sim \exp (-\Lambda x) \) in \( x \)-representation.
Nonlocality of quark condensates $\lambda_q^2 = 0.42(8) \text{ GeV}^2$ from lattice data of Pisa group in comparison with local limit.

Even at $|z| \approx 0.5 \text{ fm}$ nonlocality is quite important!
Basis condensates

Bilocal quark-anti quark condensate ($A_0 = 2\alpha_s \pi \langle \bar{q}q \rangle^2 / 81$):

$$\langle \bar{q}_A(0)q_B(x) \rangle = \frac{1}{4} \int_0^\infty \left\{ \delta_{BA} \langle \bar{q}q \rangle f_S(\alpha) - iA_0 \hat{x}_{BA} f_V(\alpha) \right\} e^{\alpha x^2 / 4} d\alpha.$$  

Quark-gluon condensate in fixed-point gauge $x^\mu A_\mu(x) = 0$:

$$\langle \bar{q}_B(0)(-gA_\nu^a(y) t^a) q_A(x) \rangle = A_0 \sum_{i=1}^3 \Gamma^i_\nu(x, y)_{AB} \times$$

$$\times \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 f_i(\alpha_1, \alpha_2, \alpha_3) e^{(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 (x-y)^2) / 4},$$

where

$$\Gamma^1_\nu(x, y) = -\frac{3}{2} (\hat{y} x_\nu - \gamma_\nu (x y));$$

$$\Gamma^2_\nu(x, y) = 2 (\hat{y} y_\nu - \gamma_\nu y^2);$$

$$\Gamma^3_\nu(x, y) = i \frac{3}{2} \epsilon_{\nu \sigma xy} \gamma_5 \gamma^\sigma.$$
Bakulev, Mikhailov, Radyushkin, and Stefanis use the minimal Gaussian ansatz:

\[ f_S(\alpha) = \delta \left( \alpha - \frac{\lambda_q^2}{2} \right), \quad f_V(\alpha) = \delta' \left( \alpha - \frac{\lambda_q^2}{2} \right), \]

\[ f_i(\alpha_1, \alpha_2, \alpha_3) = \delta \left( \alpha_1 - \frac{\lambda_q^2}{2} \right) \delta \left( \alpha_2 - \frac{\lambda_q^2}{2} \right) \delta \left( \alpha_3 - \frac{\lambda_q^2}{2} \right) \]

There is one parameter \( \lambda_q^2 = 0.4 - 0.5 \) GeV\(^2\).

The transition to local condensate case is \( \lambda_q^2 \to 0 \).

This model provides the DAs and FFs of light mesons in good agreement with experimental data.

**Problems:**

- QCD equations of motion are violated
- Vector current correlator is not transverse
  \( \Rightarrow \) gauge invariance is broken
QCD equation of motion for condensates

From Dirac equation for massless quark $(\hat{A}_\mu(x) \equiv A_\mu^a(x) t^a)$:

$$(\partial_\mu - ig\hat{A}_\mu(x))\gamma_\mu q(x) = 0,$$

one can obtain QCD equation of motion for splitted vector quark current

$$(\partial_\mu - ig\hat{A}_\mu(x))\bar{q}(0)\gamma_\mu q(x) = 0$$

If we average it over physical QCD vacuum, then we obtain the equation for condensates:

$$\partial_\mu \langle \bar{q}(0)\gamma_\mu q(x) \rangle = i\langle \bar{q}(0) g\hat{A}_\mu(x)\gamma_\mu q(x) \rangle.$$

Minimal Gaussian ansatz does not satisfy this equation.
Improved Gaussian model

We modify functions $f_i$ by introducing new parameters:

$$f_S(\alpha) = \delta (\alpha - \Lambda_S), \quad f_V(\alpha) = \delta'(\alpha - \Lambda_V),$$

$$f_{i\text{imp}}(\alpha_1, \alpha_2, \alpha_3) = (1 + X_i \partial_x + Y_i \partial_y + Y_i \partial_z) \delta (\alpha_1 - x\Lambda_V) \delta (\alpha_2 - y\Lambda_V) \delta (\alpha_3 - z\Lambda_V).$$

What does it give?:

- If these conditions $12 (X_2 + Y_2) - 9 (X_1 + Y_1) = 1, \ x + y = 1$, are fulfilled than QCD equations of motion are satisfied;
- We minimize nontransversity of polarization operator by special choice of model parameters;
- Using improved model causes changing results (pion DA, pion em. FF) but on values that are smaller than theoretical errors.
The pion DA parameterizes this matrix element:

\[
\langle 0 | \bar{d}(z) \gamma_\nu \gamma_5 [z, 0] u(0) | \pi(P) \rangle \bigg|_{z^2=0} = i f_\pi P_\nu \int_0^1 dx \ e^{ix(zP)} \varphi_\pi(x, \mu^2).
\]

where the path-ordered exponential

\[
[z, 0] = \mathcal{P} \exp \left[ i g \int_0^z t^\alpha A_\mu^\alpha(y) dy^\mu \right],
\]

i.e., the light-like gauge link, ensures the gauge invariance.

Pion DA describes the transition of a physical pion into two valence quarks, separated at light cone.
The pion DA parameterizes this matrix element:

\[
\left. \langle 0 | \bar{d}(z)\gamma_\nu\gamma_5 [z, 0] u(0) | \pi(P) \rangle \right|_{z^2=0} = i f_\pi P_\nu \int_0^1 dx \ e^{ix(zP)} \varphi_\pi(x, \mu^2).
\]

Distribution amplitudes are nonperturbative quantities to be derived from

- QCD SR [CZ 1984],
- instanton-vacuum approaches, e.g.
  [Dorokhov et al. 2000; Polyakov et al. 1998, 2009]
- Lattice QCD, [Braun et al. 2006; Donnellan et al. 2007]

DA evolves with \( \mu^2 \) according to ERBL equation in pQCD.
The pion DA parameterizes this matrix element:

\[
\langle 0| \bar{d}(z)\gamma_\nu\gamma_5[\gamma_\mu, 0]u(0) |\pi(P)\rangle \bigg|_{z^2=0} = i f_{\pi\nu} \int_0^1 dx e^{ix(zP)} \varphi_\pi(x, \mu^2).
\]

There are numbers of models for pion DA on a market. We could qualitatively collect them in two groups by their behavior at the end-point region \(x = 0\):

**end-point suppressed** and **end-point enhanced** pion DAs.
QCD SR for pion DA

QCD SR technique for correlator of two axial current leads to SR for $\pi$-DA $\varphi_\pi(x)$:

$$f_\pi^2 \varphi_\pi(x) + f_{A_1}^2 \varphi_{A_1}(x) e^{-m_{A_1}^2/M^2} = \int_0^{s_0} \rho_{pert}(s, x) e^{-s/M^2} ds + \Phi_{npert}(x, M^2),$$

where $\Phi_{npert} = \Phi_{4Q} + \Phi_T + \Phi_V + \Phi_G$, $M^2$ – Borel parameter, $\rho_{pert}$ – pert. spec. density.

The largest nonperturbative term:

$$\Phi_{4Q} \sim x\theta(\Delta - x) \xrightarrow{\text{loc. lim}} \Phi_{4Q}^{\text{loc}} \sim \delta(x),$$

is defined by scalar quark condensate, where $\Delta = \lambda_q^2/M^2 \in [0.01, 0.3]$.

Since nonperturbative contribution has singularities $(x\delta'(\Delta - x), \delta(\Delta - x))$, we should study integral characteristics of $\pi$-DA in order to take into account all condensates and reduce model dependence.

Exception is end-point region where only 4-quark condensate $\Phi_{4Q}$ contributes without any singularities.
Integral characteristics of pion DA

Moments: $\langle \xi^{2N} \rangle \equiv \int_0^1 dx \varphi_\pi(x)(2x - 1)^{2N}$, $\langle x^{-1} \rangle \equiv \int_0^1 dx \varphi_\pi(x)x^{-1}$.

<table>
<thead>
<tr>
<th></th>
<th>$\langle \xi^0 \rangle$</th>
<th>LO local cond.</th>
<th>$f_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVZ</td>
<td>$\langle \xi^{2N} \rangle$, $N = 0, 1$</td>
<td>LO local cond.</td>
<td>$f_\pi, a_2$</td>
</tr>
<tr>
<td>CZ</td>
<td>$\langle \xi^{2N} \rangle$, $N = 0, 1, \ldots, 5$</td>
<td>NLO nonlocal cond.</td>
<td>$f_\pi, a_2, a_4, \langle x^{-1} \rangle$</td>
</tr>
<tr>
<td>BMS</td>
<td>$\langle \xi^{2N} \rangle$, $N = 0, 1, \ldots, 5$</td>
<td>NLO nonlocal cond.</td>
<td>$f_\pi, a_2, a_4, \langle x^{-1} \rangle$</td>
</tr>
<tr>
<td>Here</td>
<td>$<a href="x">D^{(\nu)}\varphi_\pi</a>$</td>
<td>NLO nonlocal cond.</td>
<td>$\varphi'_\pi(0)$</td>
</tr>
</tbody>
</table>

Pion DA in a form of Gegenbauer expansion:

$$\varphi_\pi(x; \mu^2) = 6x\bar{x}\left[1 + a_2 C_2^{3/2}(2x - 1) + a_4 C_4^{3/2}(2x - 1) + \ldots\right]$$

We extract the $(a_2, a_4)$ Gegenbauer coefficients from QCD SRs on the Moments Region for $(a_2, a_4)$ of the pion DA for improved model (solid line) in comparison with minimal result: BMS model (○) and bunch (dashed line).
QCD SR for $\varphi'_\pi(0)$ in Gaussian model

By differentiating QCD SR for pion DA at $x = 0$. We arrive at SR for $\varphi'_\pi(0)$

$$f^2_\pi \varphi'_\pi(0) = \frac{3}{2\pi^2} M^2 \left(1 - e^{-s_0/M^2}\right) - f^2_{A_1} \varphi'_{A_1}(0) e^{-m^2_{A_1}/M^2} + \frac{144\pi\alpha_s}{81} \langle \bar{q}q \rangle^2 \Phi' ,$$

where only 4-quark condensate contribution survives.

Nonperturbative term mainly defined by scalar-quark condensate at large and moderate distances

$$\Phi' = \int_0^\infty d\alpha f_S(\alpha) \frac{\langle \bar{q}q \rangle}{\alpha^2} = \langle \bar{q}q \rangle^{-1} \int_0^\infty z^2 \langle \bar{q}(0)q(z) \rangle dz^2 .$$

Simplest assumption for scalar condensate model $f_S(\alpha) = \delta(\alpha - \lambda^2_q/2)$ leads to Gaussian behavior $\sim \exp(\lambda^2_q x^2/8)$ of coordinate dependence and to simple expression for nonperturbative contribution to SR:

$$\Phi' \rightarrow \Phi'_{\text{Gauss}} = 4/\lambda^4_q .$$

Then QCD SR result is $\varphi'_\pi(0) = 5.3(5)$, where nonlocality parameter $\lambda^2_q = 0.4\,\text{GeV}^2$ was used.
QCD SR for $\varphi'_\pi(0)$ with smooth NLC

There is an indication from heavy-quark effective theory [Radyushkin 91] that in reality quark-virtuality distribution $f_S$ should be parameterized in a different way as to ensure that scalar condensate decreases exponentially at large distances.

$$\langle \bar{q}(0)q(z) \rangle \sim |z|^{-(2n+1)/2} e^{-\Lambda|z|}.$$  

This could be realized by model:

$$f_S(\alpha; \Lambda, n, \sigma) \sim \alpha^{n-1} e^{-\Lambda^2/\alpha - \alpha \sigma^2}.$$  

Analysis of SR for the heavy-light meson, obtained in heavy quark effective theory, leads to values $\Lambda = 0.45$ GeV and $n = 1$. For these parameters we get $\varphi'_\pi(0) = 7.0(7)$ (black point in Fig.).

Slower decay at large distances, causes an increase of the pion DA slope $\varphi'_\pi(0)$. 

![Graph showing the dependence of $\varphi'_\pi(0)$ on $n$ for different values of $\Lambda$.](image-url)
Comparison of results with pion DA models

<table>
<thead>
<tr>
<th>Approach</th>
<th>$<a href="0.5">D^{(3)}\varphi_\pi</a>$</th>
<th>$\varphi'_\pi(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integral LO QCD SR</td>
<td>$4.7 \pm 0.5$</td>
<td>$5.5 \pm 1.5$</td>
</tr>
<tr>
<td>Differential LO QCD SR, Gaussian decay of NLC</td>
<td>—</td>
<td>$5.3 \pm 0.5$</td>
</tr>
<tr>
<td>Differential LO QCD SR, exponential decay of NLC</td>
<td>—</td>
<td>$7.0 \pm 0.7$</td>
</tr>
</tbody>
</table>

$$[D^{(\nu+2)}\varphi_\pi](x) = \frac{1}{x} \int_0^x dy \varphi_\pi(y) \frac{(\log x/y)^\nu}{y\Gamma(1 + \nu)}$$

<table>
<thead>
<tr>
<th>Curve</th>
<th>Model</th>
<th>$<a href="0.5">D^{(3)}\varphi_\pi</a>$</th>
<th>$\varphi'_\pi(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BMS DA</td>
<td>$5.7 \pm 1.0$</td>
<td>$1.7 \pm 5.3$</td>
</tr>
<tr>
<td></td>
<td>Asy DA</td>
<td>$5.25$</td>
<td>$6$</td>
</tr>
<tr>
<td></td>
<td>CZ DA</td>
<td>$15.1$</td>
<td>$26.2$</td>
</tr>
<tr>
<td></td>
<td>$\sim x^{0.1}$</td>
<td>$227$</td>
<td>$\gg 6$</td>
</tr>
<tr>
<td></td>
<td>[WH10]</td>
<td>$14$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
**Definition of pion Form Factor**

Pion FF $F_\pi$ is defined by the matrix element

$$\langle \pi^+(p_2)|J_\mu(0)|\pi^+(p_1)\rangle = (p_1 + p_2)_\mu F_\pi(Q^2),$$

where $J_\mu$ is the electromagnetic current, 

$$(p_2 - p_1)^2 = q^2 \equiv -Q^2$$

is the photon virtuality, and pion FF is normalized to $F_\pi(0) = 1$.

We are interested in space-like region $Q^2 > 0$.

At asymptotically large $Q^2 \gtrsim 20$ GeV$^2$, the pQCD factorization gives the pion FF

$$F_\pi(Q^2) = \frac{8 \pi \alpha_s(Q^2)f_\pi^2}{9 Q^2} \left| \int_0^1 \varphi_\pi(x, Q^2) \frac{dx}{x} \right|^2,$$

in terms of the pion DA $\varphi_\pi(x, Q^2)$ of the leading twist.
Pion form factor from AAV correlator

For intermediate momentum transfer $1 \text{ GeV}^2 \lesssim Q^2 \lesssim 20 \text{ GeV}^2$ one can use QCD SR technique via Axial-Axial-Vector correlator:

$$\int \int d^4x \, d^4y \, e^{i(qx-p_2y)} \langle 0| T \left[ J^+_{5\beta}(y) J^\mu(x) J^\alpha_{5\alpha}(0) \right] |0 \rangle$$

where EM current $J^\mu(x) = e_u \bar{u}(x) \gamma^\mu u(x) + e_d \bar{d}(x) \gamma^\mu d(x)$ and axial-vector current: $J^\alpha_{5\alpha}(x) = \bar{d}(x) \gamma_5 \gamma^\alpha u(x)$.
Diagramms for AAV-correlator

Perturbative LO term

Nesterenko & Radyushkin
\textcircled{Ioffe & Smilga 1982}

Perturbative NLO terms

Braguta & Onishchenko 2004

Nonperturbative terms

Nesterenko & Radyushkin
\textcircled{Ioffe & Smilga 1982}
local condensates
Diagramms for AAV-correlator

Perturbative LO term

Nesterenko & Radyushkin
Ioffe & Smilga 1982

Perturbative NLO terms

Braguta & Onishchenko 2004

Nonperturbative terms

Bakulev & Radyushkin 1991

nonlocal condensates
QCD SR with local condensates

The Borel SR for the pion FF based on three-point AAV correlator:

\[ f_\pi^2 F_\pi(Q^2) = \int_0^{s_0} ds_1 \int ds_2 \rho(s_1, s_2, Q^2) e^{-(s_1+s_2)/M^2} + \Phi_{\text{nonpert}}(Q^2, M^2). \]

\[ \Phi_{\text{nonpert}}(Q^2, M^2) = \frac{\langle \alpha_s G G \rangle}{12 \pi M^2} + \frac{208 \alpha_s \pi \langle \bar{q}q \rangle^2}{81M^4} \left( 1 + \frac{2 Q^2}{13 M^2} \right). \]

Wrong scale behavior of nonperturbative terms at large \( Q^2 \).

SR becomes unstable for \( Q^2 > 3 \text{ GeV}^2 \).
QCD SR with nonlocal condensates

The difference between local ($\lambda_q^2 \to 0$) and nonlocal case could be shown on an example of the vector quark condensate contribution to three-point AAV correlator:

$$
Q^2 = \frac{16 \alpha_s \pi \langle \bar{q} q \rangle^2}{81 M^4} \left( 2 + \frac{Q^2}{M^2 - \lambda_q^2} \right) \exp \left[ -\frac{Q^2 \lambda_q^2}{2 M^2 (M^2 - \lambda_q^2)} \right]
$$

$$
\sim \frac{1}{M^4} \left( 2 + \frac{Q^2}{M^2} \right) - \frac{\lambda_q^2}{2} \frac{Q^4}{M^{10}} + \ldots \text{ for } \lambda_q^2 \to 0.
$$

- The taking into account the nonlocality $\lambda_q^2$ expands the admissible region in QCD SR up to $Q^2 \sim 7$.

- For momentum $Q^2 < 7$ the results is weekly depending on modeling.
QCD SR with nonlocal condensates

The Borel SR for the pion FF based on three-point AAV correlator:

\[
f_\pi^2 F_\pi(Q^2) = \int_0^{s_0} ds_1 \int_0 ds_2 \, \rho(s_1, s_2, Q^2) \, e^{-(s_1+s_2)/M^2} + \Phi_{\text{OPE}}(Q^2, M^2).
\]

<table>
<thead>
<tr>
<th>Approach</th>
<th>Acc</th>
<th>Condensates</th>
<th>(Q^2)-behavior of (\Phi_{\text{OPE}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard SR</td>
<td>LO</td>
<td>local</td>
<td>(c_1 + Q^2/M^2) where (c_i \neq f(Q^2))</td>
</tr>
<tr>
<td>SR with NLC</td>
<td>LO</td>
<td>local + nonlocal</td>
<td>((c_2 + \frac{Q^2}{M^2}) \left( e^{-c_3 Q^2 \lambda_q^2/M^4} + c_4 \right))</td>
</tr>
<tr>
<td>LD SR ((M^2 \to \infty))</td>
<td>NLO</td>
<td>NO</td>
<td>0</td>
</tr>
<tr>
<td>Here</td>
<td>NLO</td>
<td>nonlocal</td>
<td>((c_5 + Q^2/M^2) , e^{-c_6 Q^2 \lambda_q^2/M^4})</td>
</tr>
</tbody>
</table>

Using nonlocal condensates improves \(Q^2\) behavior of OPE and as a result widens region of applicability up to \(Q^2 \approx 10\,\text{GeV}^2\).

We use model-independent expression for \(\Phi_{\text{OPE}}\)-term obtained by Bakulev&Radyushkin, but significantly different model of condensate’s nonlocality.
Pion FF in QCD SR with nonlocal condensates in minimal and improved models in comparison with lattice simulations, experimental results and other approaches.

We wait for the data up to 6 GeV$^2$ from JLab 12 GeV$^2$ Upgrade!
Pion FF in QCD SR with nonlocal condensates in minimal and improved models in comparison with lattice simulations, experimental results and other approaches. We wait for the data up to 6 GeV$^2$ from JLab 12 GeV$^2$ Upgrade!
Pion FF in QCD SR with nonlocal condensates in minimal and improved models in comparison with lattice simulations, experimental results and other approaches.

We wait for the data up to 6 GeV$^2$ from JLab 12 GeV$^2$ Upgrade!
Pion FF in QCD SR with nonlocal condensates in minimal and improved models in comparison with lattice simulations, experimental results and other approaches.

We wait for the data up to 6 GeV$^2$ from JLab 12 GeV$^2$ Upgrade!
Pion FF in QCD SR with nonlocal condensates in minimal and improved models in comparison with lattice simulations, experimental results and other approaches.

We wait for the data up to 6 GeV$^2$ from JLab 12 GeV$^2$ Upgrade!
“Factorization” \( \gamma^*(q_1)\gamma^*(q_2) \rightarrow \pi^0(P) \) in pQCD

\[
\int d^4 x e^{-i q_1 \cdot z} \langle \pi^0(P) | T \{ j_\mu(z) j_\nu(0) \} | 0 \rangle = i \epsilon_{\mu \nu \alpha \beta} q_1^\alpha q_2^\beta \cdot F_{\gamma^* \gamma^* \pi}(Q^2, q^2),
\]

where \( -q_1^2 = Q^2 > 0, \ -q_2^2 = q^2 \geq 0 \)

Collinear factorization at \( Q^2, q^2 \gg \) (hadron scale \( \sim m_\rho^2 \))

\[
F_{\gamma^* \gamma^* \pi}(Q^2, q^2) = T(Q^2, q^2, \mu_F^2; x) \otimes \varphi_\pi(x; \mu_F^2) + O(\frac{1}{Q^4}),
\]

where \( \mu_F^2 \) – boundary between large scale and hadronic one.

\[
F_{\gamma^* \gamma^* \pi}(Q^2, q^2) = \frac{\sqrt{2}}{3} f_\pi \int_0^1 dx \frac{1}{Q^2 x + q^2 x} \varphi_\pi(x)
\]

\[
Q^2 F_{\gamma^* \gamma^* \pi}(Q^2, q^2 \rightarrow 0) = \frac{\sqrt{2}}{3} f_\pi \int_0^1 \frac{dx}{x} \varphi_\pi(x) \equiv \frac{\sqrt{2}}{3} f_\pi \langle x^{-1} \rangle_\pi
\]
\[ \gamma \gamma \rightarrow \pi : \text{Why Light-Cone Sum Rules?} \]

For \( Q^2 \gg m^2_{\rho}, \ q^2 \ll m^2_{\rho} \) pQCD factorization valid only in leading twist and higher twists are of importance [Radyushkin–Ruskov, NPB (1996)].

Reason: if \( q^2 \rightarrow 0 \) one needs to take into account interaction of real photon at long distances \( \sim O(1/\sqrt{q^2}) \)

\[ Q^2 \gg m^2_{\rho} \quad \rightarrow \quad \pi(P) \]

\[ q^2 \gg m^2_{\rho} \]

pQCD is OK

\[ Q^2 \gg m^2_{\rho} \quad \rightarrow \quad \pi(P) \]

\[ q^2 \approx 0 \]

LCSRs should be applied
\[\gamma^*\gamma \rightarrow \pi: \text{Why Light-Cone Sum Rules?}\]

- For \(Q^2 \gg m_\rho^2, \quad q^2 \ll m_\rho^2\) pQCD factorization valid only in leading twist and higher twists are of importance [Radyushkin–Ruskov, NPB (1996)].

- Reason: if \(q^2 \rightarrow 0\) one needs to take into account interaction of real photon at long distances \(\sim O(1/\sqrt{q^2})\)

- To account for long-distance effects in pQCD one needs for light-cone \textit{DA} of real photon
Khodjamirian [EJPC (1999)]: LCSR effectively accounts for long-distances effects of real photon using quark-hadron duality in vector channel and dispersion relation in $q^2$

\[ F_{\gamma\gamma^*\pi}(Q^2, q^2) = \int_{0}^{s_0} \frac{\rho^{\text{PT}}(Q^2, s)}{m_\rho^2 + q^2} e^{(m_\rho^2 - s)/M^2} ds + \int_{s_0}^{\infty} \frac{\rho^{\text{PT}}(Q^2, s)}{s + q^2} ds, \]

where $s_0 \approx 1.5 \text{ GeV}^2$ – effective threshold in vector channel, $M^2$ – Borel parameter ($0.5 – 0.9 \text{ GeV}^2$).

Real-photon limit $q^2 \rightarrow 0$ can be easily done.

Spectral density is defined by Im-part of FF for two virtual photons:

\[ \rho^{\text{PT}}(Q^2, s) = \text{Im} F^{\text{PT}}_{\gamma\gamma^*\pi}(Q^2, -s - i\epsilon) = Tw-2 + Tw-4 + Tw-6 + \ldots, \]

where twists contributions given in a form of convolution with pion DA:

\[ Tw-2 \sim (T_{\text{LO}} + T_{\text{NLO}} + T_{\text{NNLO}} + \ldots) \otimes \phi_{T^2}^{\pi}(x). \]
Main Ingredients of Spectral Density

- **LO Spectral Density**, **Tw-4** term — Khodjamirian [EJPC (1999)]
- **NLO** Spectral Density — in [Mikhailov & Stefanis (2009)]
- **NNLO**_{\beta_0} Spectral Density — in [M&S (2009)]
- **Tw-6** contribution — in [Agaev et al. – PRD83 (2011) 0540020]

Terms of Pion-Photon FF at \( Q^2 = 8 \text{ GeV}^2 \)

- Result is dominated by Hard Part of Twist-2 LO contribution.
- Twist-6 contribution is taken into account together with **NNLO**_{\beta_0} one — they have close absolute values and opposite signs.

Blue - negative terms
Red - positive terms

Frontiers in QCD, INT, 2011
Parameters of LC SR

From PDG:
- $\alpha_s(m_Z^2)$
- Masses $m_\rho, m_\omega$
- Decay Widths $\Gamma_\rho, \Gamma_\omega$

From QCD SR:
- Borel parameter $M_{\text{LCSR}}^2$
- Vector Chan. Threshold $s_0$
- Twist-4 $\delta^2 \pm 20\%$
- Twist-6 ($\alpha_s \langle \bar{q}q \rangle$)

Light-Cone Sum Rules:
$$FF = (\text{LO} + \text{NLO}) \otimes (\pi-\text{DA}) + \text{Tw-4} + (\text{NNLO}_\beta + \text{Tw-6})$$

- $\pi$-DA model
- Data on FF
- FF Prediction
- Fitting $\pi$-DA ($a_n$)
Feynman diagram for $e^+e^- \rightarrow e^+e^-\pi^0$

One of the most accurate data on exclusive reactions is data on transition FF $F_{\gamma^*\gamma^*\pi^0}(q_1^2, q_2^2)$ provided by series of experiments $e^+e^- \rightarrow e^+e^-\pi^0$ with $q_2^2 \approx 0$.

**CELLO** (1991) $0.7 - 2.2$ GeV$^2$

**CLEO** (1998) $1.6 - 8.0$ GeV$^2$

**BaBar** (2009) $4 - 40$ GeV$^2$
Pion-gamma FF vs Experimental Data

Comparison with all data: **CELLO**, CLEO and **BaBar**

- **BMS bunch** describes very good all data for $Q^2 \leq 9$ GeV$^2$.
Pion-gamma FF vs Experimental Data

Comparison with all data: CELLO, CLEO and BaBar

\[ Q^2 F(Q^2) \ [\text{GeV}^2] \]

<table>
<thead>
<tr>
<th>curve</th>
<th>DA</th>
</tr>
</thead>
<tbody>
<tr>
<td>----</td>
<td>Asymp.QCD</td>
</tr>
<tr>
<td>BMS bunch</td>
<td>GR-PRD77-115024</td>
</tr>
</tbody>
</table>

- **BMS bunch** describes very good all data for \( Q^2 \leq 9 \text{ GeV}^2 \).
- Note presented by BaBar rotation of \( \gamma^* \gamma \rightarrow \eta, \eta' \) and \( e^+ e^- \rightarrow \gamma \eta, \gamma \eta' \) data (1101.1142[hep-ex]) to pion FF using \( \eta - \eta' \) mixing scheme agrees with **BMS strip**!
Comparison with all data: **CELLO**, CLEO and **BaBar**

---

- **BMS bunch** describes very good all data for $Q^2 \leq 9$ GeV$^2$.
- Note presented by BaBar rotation of $\gamma^*\gamma \rightarrow \eta, \eta'$ and $e^+e^- \rightarrow \gamma\eta, \gamma\eta'$ data (1101.1142[hep-ex]) to pion FF using $\eta - \eta'$ mixing scheme agrees with **BMS strip**!
- **ABOP models** are in between two sets of BaBar data.

---

Frontiers in QCD, INT, 2011
We fitted experim. data on $\pi\gamma$ TFF by varying Gegenbauer coefficients of Pion DA. Two sets of experim. data ($1 - 9$ GeV$^2$ & $1 - 40$ GeV$^2$) were analyzed to show the influence of BaBar Data on Pion DA. To have an agreement with all data at the level $\chi^2_{ndf} \approx 1$ we need to take at least 3 terms of pion DA Gegenbauer expansion with corresponding coefficients $a_2, a_4, a_6$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{Fitting $Q^2 F(Q^2)$ for $Q^2 [\text{GeV}^2]$ values between $1$ and $20$.}
\end{figure}
NLC SR Results vs 3D Constraints

BMPS [arXiv:1105.2753 [hep-ph]]: 3D $1\sigma$-error ellipsoid at $\mu_{SY} = 2.4$ GeV scale without theoretical $\Delta\delta_{tw4}^2$ uncertainties.

$1 - 9$ GeV$^2$ Data

- $2D$ projection of $1\sigma$-error ellipsoid
- $\nabla$ $\Leftrightarrow$ $\chi_{ndf}^2 \approx 0.4$
- $\times$ $\Leftrightarrow$ BMS model with $\chi_{ndf}^2 \approx 0.5$

Best-fit = $(0.17, -0.14, 0.12 \pm 0.14)$
BMS = $(0.14, -0.09)$

Good agreement with all data at $Q^2 \leq 9$ GeV$^2$
At 68.3% CL we have good intersection $2D \cap 3D \cap 4D \neq \emptyset$
NLC SR Results vs 3D Constraints

BMPS [arXiv:1105.2753 [hep-ph]]: 3D $1\sigma$-error ellipsoid at $\mu_{\text{SY}} = 2.4$ GeV scale without theoretical $\Delta\delta_{\text{tw4}}$ uncertainties.

$1 - 40 \text{ GeV}^2$ Data

- $2\sigma$ projection of $1\sigma$-error ellipsoid
- $\chi^2_{\text{ndf}} \approx 1.0$
- BMS model with $\chi^2_{\text{ndf}} \approx 3.1$

Good agreement with all data at $Q^2 \leq 9 \text{ GeV}^2$

At $68.3\%$ CL we have good intersection $2D \cap 3D \cap 4D \neq \emptyset$

Frontiers in QCD, INT, 2011
NLC SR Results vs 2D Constraints

NLC-bunch and lattice prediction at $\mu_{\text{SY}} = 2.4$ GeV scale with $\Delta \delta_{\text{tw4}}$ error

DAs: $\blacklozenge \leftrightarrow \text{Asymp.}$, $\blacktriangle \leftrightarrow \text{ABOP-3}$, $\blacklozenge \leftrightarrow \text{BMS}$, $\blacksquare \leftrightarrow \text{CZ}$

Lattice’10 estimate of $a_2$ are shown by vertical lines.

BMS bunch agrees well with the lattice data

1 – 9 GeV$^2$ Data
NLC SR Results vs 2D Constraints

2D-Analysis of the data at $\mu_{SY} = 2.4$ GeV scale with $\Delta \delta^2_{tw4}$ error

DAs: ◆ ⇔ Asymp., ▲ ⇔ ABOP-3, ✗ ⇔ BMS, ■ ⇔ CZ

Lattice’10 estimate of $a_2$ are shown by vertical lines.

BMS bunch agrees well with the lattice data

BMS bunch has better agreement with data up 9 GeV$^2$ than with CLEO data only.
NLC SR Results vs 2D Constraints

2D cut of 3D ellipsoid of the data analysis at $\mu_{SY} = 2.4$ GeV scale with $\Delta \delta_{tw4}^2$ error

DAs: ◆ ⇔ Asymp., ▲ ⇔ ABOP-3, ✗ ⇔ BMS, ■ ⇔ CZ

Lattice’10 estimate of $a_2$ are shown by vertical lines.

BMS bunch agrees well with the lattice data

BMS bunch has better agreement with data up to $9 \text{ GeV}^2$ than with CLEO data only.
NLC SR Results vs 2D Constraints

BMPS [arXiv:1105.2753 [hep-ph]]: 2D $1\sigma$-error ellipses at $\mu_{SY} = 2.4$ GeV scale with $\Delta \delta_{tw4}^2$ error

DAs: ◆ ⇔ Asymp., ▲ ⇔ ABOP-3, ✗ ⇔ BMS, ■ ⇔ CZ

Lattice'10 estimate of $a_2$ are shown by vertical lines.

Bad agreement with 2D $1\sigma$-error ellipse

no cross-section with $a_6 = 0$ plane.
**NLC SR Results vs 2D Constraints**

**BMPS [arXiv:1105.2753 [hep-ph]]**: 2D $1\sigma$-error ellipses at $\mu_{\text{SY}} = 2.4$ GeV scale with $\Delta \delta^2_{\text{tw4}}$ error

DAs: ◆ $\Leftrightarrow$ Asymp., ▲ $\Leftrightarrow$ ABOP-3, × $\Leftrightarrow$ BMS, ■ $\Leftrightarrow$ CZ

**Lattice’10** estimate of $a_2$ are shown by vertical lines.

---

![Diagram](image.png)

- **Bad agreement** with 2D $1\sigma$-error ellipse
- **no cross-section** with $a_6 = 0$ plane.

---

1 - 40 GeV$^2$ Data

- $\Leftrightarrow$ 2D $1\sigma$-error ellipse
- $\Leftrightarrow$ 2D-Proj. 3D-ellipsoid
Data fit of pion DA vs QCD SR

→ BMS, → 1 − 9 GeV², → 1 − 40 GeV² at \( \mu_{\text{SY}} = 2.4 \) GeV scale

- BMS bunch agrees well with 1 − 9 GeV²
- New BaBar data does not agree with BMS bunch based on NLC QCD SR.
- Both data sets does not match each other only at the end point region.
- 1 − 9 GeV² based DA and 1 − 40 GeV² based DA separated near origin.
- High BaBar data demands the end-point enhanced behavior from pion DA.
# Comparison of fit with pion DA models

<table>
<thead>
<tr>
<th>Model/Fit</th>
<th>Values of $a_n$</th>
<th>$\chi^2_{\text{ndf}}$ (1 – 9 GeV$^2$)</th>
<th>$\chi^2_{\text{ndf}}$ (1 – 40 GeV$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_2$, $a_4$, $a_6$ fit</td>
<td>(0.18, $-0.17$, 0.31)</td>
<td>0.4</td>
<td>1.0</td>
</tr>
<tr>
<td>NLC QCD SR, <strong>BMS</strong></td>
<td>(0.141, $-0.089$)</td>
<td>0.5</td>
<td>3.1</td>
</tr>
<tr>
<td>Agaev et al</td>
<td>(0.084, 0.137, 0.088)</td>
<td>$\geq 2.8$</td>
<td>$\geq 2.4$</td>
</tr>
<tr>
<td>Modif. fact. fit, Kroll</td>
<td>(0.21, 0.009)</td>
<td>3.8</td>
<td>4.4</td>
</tr>
<tr>
<td>AdS/QCD, Brodsky et al</td>
<td>0.15, 0.06, 0.03</td>
<td>2.3</td>
<td>2.8</td>
</tr>
<tr>
<td>CZ</td>
<td>(0.394)</td>
<td>32.3</td>
<td>25.5</td>
</tr>
<tr>
<td>Asympt.</td>
<td>(0, 0)</td>
<td>4.7</td>
<td>7.9</td>
</tr>
</tbody>
</table>

All $a_n$ values given at $\mu_{\text{SY}} = 2.4$ GeV scale.

- **BMS DA gives best description in LC SR of FF for momentum up to 9 GeV$^2$.**
- Result of all data fit in LC SR is far from all considered model of pion DA.
## [1 – 9] vs [1 – 40] GeV$^2$ data analyses

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BMS bunch</strong></td>
<td>Agreement</td>
<td>No!</td>
</tr>
<tr>
<td>number of harmonics $n$</td>
<td>2, 3</td>
<td>3, 4</td>
</tr>
<tr>
<td>best $\chi^2_{ndf}$</td>
<td>0.53, 0.44</td>
<td>1.0, 0.77</td>
</tr>
<tr>
<td>Slope $\varphi'_\pi(0)$</td>
<td>20.2 ± 20.9</td>
<td>48.5 ± 11.8</td>
</tr>
<tr>
<td>Slope $D^{(3)}\varphi_\pi(0.5)$</td>
<td>8.3 ± 3.2</td>
<td>12.7 ± 1.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NLC Model</th>
<th>gaussian</th>
<th>exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi'_\pi(0)$</td>
<td>5.3 ± 0.5</td>
<td>7.0 ± 0.7</td>
</tr>
<tr>
<td>$D^{(3)}\varphi_\pi(0.5)$</td>
<td>4.7 ± 0.5</td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

Slope of pion DA at the origin is limited by “speed” of quark condensate decay at large distances. Slower decay at large distances, causes an increase of the pion DA slope $\varphi'_\pi(0)$.

LO QCD sum rules with natural choices of NLC lead to behavior at the origin close to asymptotic DA and contradicting flat-type pion DAs.

Taking into account nonlocality of condensates enlarge the region of applicability of SR towards momenta as high as $10 \text{ GeV}^2$. Result on EM pion FF is in a good agreement with existing experimental data between $1 - 10 \text{ GeV}^2$.

The result from CELLO, CLEO, and BaBar data up to $9 \text{ GeV}^2$ is in good agreement with previous CLEO based fit and prefers a end-point suppressed pion DA, like BMS bunch;

Beyond $9 \text{ GeV}^2$, the best fit to all data on $F_{\gamma^* \gamma \rightarrow \pi}(Q^2)$ including higher BaBar points requires a sizeable coefficient $a_6$, while the $a_2$ and $a_4$ remain the same. All data fit prefers a end-point enhanced pion DA.