Rapidity Renormalization Group
... and its Applications

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Outline

1. Introduction
   - Event Shape, Angularities, and Jet Broadening

2. Soft Collinear Effective Field Theory (SCET)

3. $\eta$-Regulator and $\nu$-Renormalization Group

4. Numerics and Data

5. Other Applications
   - Higgs $p_T$ distribution

6. Conclusion
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A large class of event shape observables can be written in the form of

\[ e(X) = \frac{1}{Q} \sum_{i \in X} |p_{i,\perp}| f_e(\eta_i), \]

where rapidity \( \eta = \frac{1}{2} \log \left( \frac{E+p_\parallel}{E-p_\parallel} \right) \)

- **Thrust**

\[ \tau = 1 - T = 1 - \frac{1}{Q} \max_{\hat{t}} \left[ \sum_i |\hat{t} \cdot p_i| \right] = \frac{1}{\sqrt{s}} \sum_{i \in X} |p_{i,\perp}| e^{-|\eta_i|} \]

- **C-parameter**

\[ C = \frac{1}{\sqrt{s}} \sum_{i \in X} |p_{i,\perp}| \frac{3}{\cosh \eta_i} \]

- **Jet Broadening**

\[ B_T = \frac{1}{2\sqrt{s}} \sum_{i \in X} |p_{i,\perp}| \]
Angularities
... as a subclass

Berger, Kucs, Sterman, 03

\[ \tau_a = \frac{1}{\sqrt{s}} \sum_{i \in X} E_i (\sin \theta_i)^a (1 - |\cos \theta_i|)^{1-a} \]

\[ = \frac{1}{\sqrt{s}} \sum_{i \in X} |p_{i,\perp}| e^{-|\eta_i|(1-a)} \]

Infrared safety: \(-\infty < a < 2\)

Factorizability: \(-\infty < a < 1?\) (Hornig, Lee, Ovanesyan)
SCET\(_a\) (Chris’ talk)

For \(a \ll 1\), \(|p_{\text{soft},\perp}|/|p_{\text{coll.},\perp}| \sim e^{-\eta_{\text{coll.}}} \ll 1\)
E.g., a=0, Thrust distribution \(\tau = 1 - T\)

For \(a \sim 1\), \(|p_{\text{soft},\perp}| \sim |p_{\text{coll.},\perp}|\).
E.g., a=1, jet broadening observable, \(B\).
Factorization of Angularities

- **Factorization Theorem in QCD**
  - Collins, Soper, Sterman, ...
  - Berger, Kucs, Sterman (03)
    Angularities for $a < 1$ been calculated to NLL/LO

- **Factorization Theorem in Traditional SCET (SCET I)**
  - Bauer, Manohar, Wise, Lee, Sterman, Becher, Schwartz, Fleming, Hoang, Mantry, Stewart, ...
  - Hornig, Lee, Ovanesyan calculated angularities in SCET to NLL/LO for $a < 1$.

- **Fail at $a=1$!**
  - Spurious divergences in each sector ($a \geq 1$), yet disappear after summing over sectors as long as $a \leq 2$.
  - Rapidity divergence

- **Other cases with divergence in similar nature?**
Soft Collinear Effective Theory (SCET I)
(Luke, Bauer, Fleming, Pirjol, Stewart)

- Describe interactions between energetic particles $E \sim Q$.
- Fluctuations, $\Lambda_{QCD}$ or other low energy scales, about light cone coordinate $n = (1, 0, 0, 1)$.

Integrate out “far offshell” degrees of freedom.
  - soft-collinear decoupling

\[ p = (p \cdot \bar{n}, p \cdot n, p_\perp) \]
\[ = (p^-, p^+, p_\perp) \]

\[ p - \sim Q \]
\[ p^+ \sim \Lambda_{QCD} \]
\[ p_\perp \sim \sqrt{Q \Lambda_{QCD}} \]
SCET degrees of freedom (modes)

\[ p^\mu = (p^-, p^+, p^\perp); \quad p^2 = p^+ p^- + p^\perp \]

- Light Cone Coordinates:
  \[ n = (1, \bar{n}) \sim (1, 0, 0, 1) \]

- Power counting parameter
  \[ \lambda \equiv \frac{\Lambda_{QCD}}{Q} \]

- Hard modes: \( p^2 \sim Q^2 \) integrated out

- \( n \)-collinear
  \[ p^\mu \sim Q(1, \lambda^2, \lambda) \]

- \( \bar{n} \)-collinear
  \[ p^\mu \sim Q(\lambda^2, 1, \lambda) \]

- Usoft (\( p^2 \sim Q^2 \lambda^4 \))
  \[ p^\mu = Q(\lambda^2, \lambda^2, \lambda^2) \]
Factorization Theorem for Angularities $\tau_{a<1}$
Bauer, Fleming, Lee, Sterman, 08

<Chis’ talk on Tuesday>

- In QCD

$$\frac{d\sigma}{de} = \frac{1}{2Q^2} \sum_X |\mathcal{M}(e^+ e^- \rightarrow X)|^2 (2\pi)^4 \delta^4(e - e(x))$$

where

$$|\mathcal{M}|(e^+ e^- \rightarrow X)|^2 = \sum_{i=V,A} L_{\mu\nu}^i \langle 0 | j_i^{\mu\dagger}(x) | X \rangle \delta(e - e(X)) \langle X | j_i^{\nu}(0) | 0 \rangle$$

and $L_{\mu\nu}^i$ is the lepton tensor, and $j_i^{\mu,\nu}$ are the currents.

- Define operator $\hat{e}$ that returns the value of an event shape for a given final state $X$, $\hat{e}|X\rangle = e(X)|X\rangle$

$$\frac{d\sigma}{de} = \frac{1}{2Q^2} \int dx e^{iq \cdot x} \sum_{i=V,A} L_{\mu\nu}^i \langle 0 | j_i^{\mu\dagger}(x) \delta(e - \hat{e}) j_i^{\nu}(0) | 0 \rangle$$
Matching onto SCET

\[ j_i^\mu (x) = \sum_n \sum_{\bar{n}} C_{n\bar{n}}(\bar{p}_n, \bar{p}_{\bar{n}}; \mu) O_{n,\bar{n}}(x; \bar{p}_n, \bar{p}_{\bar{n}}; \mu), \]

in which,

\[ O_{n,\bar{n}}(x; \bar{p}_n, \bar{p}_{\bar{n}}; \mu) = e^{i(\bar{p}_n - \bar{p}_{\bar{n}})} \bar{\chi}_{n,p_n}(x) Y_n(x) \Gamma_{i}^{\mu} \bar{Y}_{\bar{n}}(x) \chi_{\bar{n},\bar{p}_{\bar{n}}}(x) \]

Write the event shape distribution in SCET in a factorized form

\[
\frac{d\sigma}{de} = \frac{1}{6Q^2} \sum_n |C_{n\bar{n}}(\bar{p}_n, \bar{p}_{\bar{n}}; \mu)|^2 \int dx \int d\bar{e}_n d\bar{e}_{\bar{n}} d\bar{e}_s \delta(e - e_n - e_{\bar{n}} - e_s) \frac{1}{N_C^2} \]

\[
\times \text{Tr}\langle 0 | \chi_{n,Q}(x)^\alpha \delta(e_n - \hat{e}_n) \bar{\chi}_{n,Q}(0)^\gamma | 0 \rangle \text{Tr}\langle 0 | \bar{\chi}_{n,\bar{Q}}(x)^\beta \delta(e_{\bar{n}} - \hat{e}_{\bar{n}}) \chi_{n,\bar{Q}}(0)^\delta | 0 \rangle \]

\[
\times \text{Tr}\langle 0 | \bar{Y}_{\bar{n}}^\dagger (x) Y_n^\dagger (x) \delta(e_s - \hat{e}_s) Y_n \bar{Y}_{\bar{n}}(0) | 0 \rangle \sum_{i=V,A} L_i \left( (\bar{\Gamma}_i^{\mu}) \right)_{\alpha\beta} (\Gamma_{i\mu})_{\gamma\delta} \]

Final result for differential event shape distribution

\[
\frac{1}{\sigma_0} \frac{d\sigma}{de} = \frac{d\sigma}{de} = H(s; \mu) \int d\bar{e}_n d\bar{e}_{\bar{n}} d\bar{e}_s \delta(e - e_n - e_{\bar{n}} - e_s) J_n(e_n; \mu) J_{\bar{n}}(e_{\bar{n}}; \mu) S(e_s; \mu) \]
[Not Quite] Back-to-Back Jets in SCET

When $a \rightarrow 1$

$$\tau_a = \frac{1}{\sqrt{s}} \sum_{i \in X} |p_{i,\perp}| \ e^{-|\eta_i|(1-a)}$$

- For $a \ll 1$, $e_s \sim e_n \sim e_{\bar{n}}$, but $|p_{s,\perp}| \ll |p_{n,\perp}| \sim |p_{\bar{n},\perp}|$ and $|\eta^s| \ll |\eta^n| \sim |\eta^{\bar{n}}|$.

  Soft radiation decoupled from the collinear radiation.

- For $a \rightarrow 1$, $\tau_{a=1} = \frac{1}{\sqrt{s}} \sum |p_{i,\perp}|$ is independent of the rapidity of the rapidity of each state. For the $e \sim e_s \sim e_n$, we need $|p_{s,\perp}| \sim |p_{n,\perp}|$.

$$S: k_s \sim (p_T, p_T, p_T)$$

$$J_{\bar{n}}: k_{\bar{n}} \sim (\sqrt{p_T^2/Q}, Q, p_T)$$

$$J_n: k_n \sim (Q, \sqrt{p_T^2/Q}, p_T)$$
SCET Modes
For $a=1$, SCET$_{II}$

- Jet Broadening Event Shape
  \[ B_T = \frac{1}{2} \tau_{a=1} = \frac{1}{2} \sum \frac{\mid \vec{k}_i \mid}{Q} \]

- Demanding $B_T \ll 1$ for dijet events

- Relevant on-shell modes with $|\vec{p}_t| \sim \lambda Q$

  - soft modes: $k_s \sim Q(\lambda, \lambda, \lambda)$
  - collinear modes: $k_n \sim Q(1, \lambda^2, \lambda)$
  - $k_{\bar{n}} \sim Q(\lambda^2, 1, \lambda)$

- Invariant massess of soft and collinear modes are on the same order $O(Q^2 \lambda^2)$

- Rapidity divergence arises as jets go soft or soft radiations go collinear since they are all on the same parabola.
Define $e \equiv \tau_{a=1}$ from now on to simplify the notification

$$\frac{d\sigma}{de} = \sigma_0 H(s) \int d\bar{e} n d\bar{e} \bar{n} d\bar{e} s \delta(e - \bar{e} n - \bar{e} \bar{n} - \bar{e} s) \times \int d\bar{p}_{1t} d\bar{p}_{2t} J_n(Q_-, \bar{e} n, \bar{p}_{1t}) J_{\bar{n}}(Q_+, \bar{e} \bar{n}, \bar{p}_{2t}) S(e_s, \bar{p}_{1t}, \bar{p}_{2t}),$$

where in covariant guages

$$(\bar{d} \equiv 2 - 2\epsilon)$$

$$J_n = \frac{2\pi \Omega_{\bar{d}}}{N_c} \langle 0 | \bar{\chi}_n \delta(\hat{P}^- - Q^-) \delta(\hat{e} - \bar{e} n) \delta(\hat{P}_{\perp} + \bar{p}_{1\perp}) \frac{\bar{\eta}}{2} \chi_n | 0 \rangle,$$

$$J_{\bar{n}} = \frac{2\pi \Omega_{\bar{d}}}{N_c} \langle 0 | \frac{\bar{\eta}}{2} \chi_{\bar{n}} \delta(\hat{P}^+ - Q^+) \delta(\hat{e} - \bar{e} \bar{n}) \delta(\hat{P}_{\perp} + \bar{p}_{2\perp}) \bar{\chi}_{\bar{n}} | 0 \rangle,$$

$$S = \rho_{1t}^{1-2\epsilon} \rho_{2t}^{1-2\epsilon} \Omega_{\bar{d}} \int \frac{d\Omega_{12}}{N_c} \times$$

$$\langle 0 | S_n^\dagger S_{\bar{n}} \delta(\hat{e} - e_s) \delta^{\bar{d}}(\hat{P}_{n\perp} - \bar{p}_{1\perp}) \delta^{\bar{d}}(\hat{P}_{\bar{n}\perp} - \bar{p}_{2\perp}) S_{n}^\dagger S_n | 0 \rangle.$$
Naive Calculation with Pure Dim-Reg

- Bare jet function:

\[ J_n(e_n, p_i, t = 0) = \frac{\alpha_s C_F}{\pi} \left( \frac{\mu^2}{Q^2 e_n^2} \right)^\epsilon \frac{1}{e_n} \int_0^1 d\bar{z} \frac{1 + (1 - \bar{z})^2}{\bar{z}}, \]

where \( z \equiv l^-/Q \), and \( l \) is the momentum of the gluon going across the cut.

- Integral ill-defined as \( z \to 0 \), the soft region.

- Leftover \( \frac{1}{\epsilon} \) divergence multiplies non-zero \( e_n \) terms that virtual diagrams, which are always proportional to \( \delta(e_n) \), cannot cancel.

- Traditional dim-reg regulating the \( \vec{k}_\perp \) part of the real radiation dose not regulate the phase space integral while \( p_T \) is fixed.
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New Regulator and $\nu$-Renormalization Group

Goal:
- Multiplicatively Renormalizable
- In the spirit of dimensional regularization
- Does not introduce new dimensionful scales in the integrants, and maintains manifest power counting in the effective theory.

$\eta$-regulator

$$W_n = \left[ \sum_{\text{perm}} \exp \left( \frac{-g}{\bar{n} \cdot \hat{P}} \left[ \frac{\left| \bar{n} \cdot \hat{P} g \right|^{-\eta}}{\nu^{-\eta}} \bar{n} \cdot A_{n,q}(0) \right] \right) \right]$$

$$S_n = \left[ \sum_{\text{perm}} \exp \left( \frac{-g}{n \cdot \hat{P}} \left[ \frac{\left| 2\hat{P}^3 g \right|^{-\eta/2}}{\nu^{-\eta/2}} n \cdot A_{s,q}(0) \right] \right) \right]$$
Regulates the $z$-momentum of each gluon coming off Wilson Line.

Preserves Exponentiation Theorems.

Preserves modes and their power counting.

Dim-reg. style evolution equations.

Does not hide soft functions, as analytic regulators do.

$\eta$ contribution spontaneously goes to zero when rapidity divergence is not present.
Zero-Bin Subtractions

No zero-bin subtraction needed with $\eta$-regulator
- Soft-bin contribution is scaleless with Rapidity Regulator $\eta$
  (as scaleless integral is 0 in pure dim-rg)
- Obtain correct IR- and UV- divergences without 0-bin subtraction
- Soft function is nonetheless non-zero
Regulator Comparison

  - Hides soft contributions (although still get fix-order matrix element correct after summing different sectors).
  - Each collinear sector has complicated regulator dependent and is meaningless before summing all sectors together...
    ⇒ breaks factorization
  - Does not exponentiation.

- **\(\Delta\)-regulator**: JC, Fuhrer, Hoang, Kelley, Manohar, 09
  - Introduces additional scales into integrals
  - Exponentiates after proper zero-bin subtraction
  - No known evolutions equation.

- **Off the light cone**: Collins, Soper, 81
  - Introduces more scales into integrals
  - Proof of exponentiation straightforward
  - Introduces gauge modes that are not appropriate by strict power counting.
  - Understood evolution equation.
Jet Function Calculation with $\eta$-regulator

$n$-direction jet function with $p_t = 0$ in Laplace space yields

$$J_n \propto \left( \frac{G^2 C_F \Omega_{2-2\epsilon}}{4(2\pi)^{3-2\epsilon}} \right) \left( \frac{\mu T}{Q} \right)^{2\epsilon} \left\{ \frac{1}{2} \frac{1 - \epsilon}{\Gamma(1 - \epsilon)} \Gamma(-2\epsilon) \right. \\
+ \frac{\Gamma(-\eta)}{\Gamma(1 - \epsilon)\Gamma(2 - \eta)} \left( \frac{\nu}{Q} \right)^{\eta} \Gamma(-2\epsilon) \right\}$$
Soft Function Calculation with $\eta$-regulator

\[
S(\tau, \vec{b}_1, \vec{b}_2) = \left( \frac{G^2 C_F \Omega_{2-2\epsilon}}{4(2\pi)^{3-2\epsilon}} \right) \left( \frac{\nu \tau}{Q} \right)^{\eta_s} \left( \frac{\mu \tau}{Q} \right)^{2\epsilon} \frac{\Gamma(-\eta_s - 2\epsilon)}{\Gamma(\eta)} \\
\times \left[ \begin{array}{c}
\binom{\frac{-\eta - 2\epsilon}{2}}{2} , \frac{1 - \eta - 2\epsilon}{2} ; 1 - \epsilon ; -\frac{b_1^2 Q^2}{\tau^2} \\
\binom{\frac{-\eta - 2\epsilon}{2}}{2} , \frac{1 - \eta - 2\epsilon}{2} ; 1 - \epsilon ; -\frac{b_2^2 Q^2}{\tau^2}
\end{array} \right]
\]

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Physics of 2-Parameter RG

\[ k^+ \]

\[ k^- \]

\[ \mu \]

\[ \nu \]
Renormalization Group Equations

- $\eta$-divergences and $\nu$-anomalous dimensions cancels when we sum up the contributions from the jet and soft functions.
- Individual $J$ and $S$ are multiplicatively renormalizable.
- $\eta$-divergence are absorbed in the renormalization constants, $Z_J, S$, such that

$$J^{(0)}_n J^{(0)}_{\bar{n}} S^{(0)} = 
\left[ Z_J(\mu, \nu) J^R_n(\mu, \nu) \right] \left[ Z_{\bar{n}}(\mu, \nu) J^R_{\bar{n}}(\mu, \nu) \right] \left[ Z_S(\mu, \nu) S^R(\mu, \nu) \right],$$

where

$$Z_J(\mu, \nu) Z_{\bar{n}}(\mu, \nu) Z_S(\mu, \nu) = Z_H^{-1}(\mu).$$
E.g. for soft function in Fourier-Laplace space

\[ Z_S(\tau, b_{1t}, b_{2t}, \mu, \nu) \]

\[ = \frac{\alpha_s C_F}{2\pi \epsilon^2} + \frac{\alpha_s C_F}{2\pi \epsilon} \ln \left( \frac{\mu^2}{\nu^2} \right) \]

\[ - \frac{\alpha_s C_F}{\pi} e^{-\epsilon \gamma_E} \frac{\Gamma(-2\epsilon)}{\Gamma(1-\epsilon)} \frac{1}{\eta} \left( \frac{\mu T e^{\gamma_E}}{Q} \right)^{2\epsilon} \times \]

\[ 2F_1 \left( -2\epsilon, \frac{1}{2} (1 - 2\epsilon); 1 - \epsilon; -\frac{b_1^2 Q^2}{\tau^2} \right) + b_1 \leftrightarrow b_2 \]
Renormalization Group Equations

- The rapidity divergences for the jet and soft functions introduce a new set of anomalous dimension \((\gamma^\nu_J, \gamma^\nu_S)\):

\[ \nu \frac{d}{d\nu} S^R(\mu, \nu) = \gamma^\nu_S S^R(\mu, \nu), \quad \nu \frac{d}{d\nu} J^R_n(\mu, \nu) = \gamma^\nu_J J^R_n(\mu, \nu) \]

Just like the traditional \(\mu\) anomalous dimension:

\[ \mu \frac{d}{d\mu} S^R(\mu, \nu) = \gamma^\mu_S S^R(\mu, \nu), \quad \mu \frac{d}{d\mu} J^R_n(\mu, \nu) = \gamma^\mu_J J^R_n(\mu, \nu) \]

- Since the cross-section is invariant under \(\mu\) and \(\nu\) variation, and that the hard function itself is free from rapidity divergence (and therefore \(\gamma^\nu_H = 0\)), we must have the relations

\[ \gamma^\mu_H + \gamma^\mu_J + \gamma^\mu_{J^\perp} + \gamma^\nu_S = 0, \quad \text{and} \quad \gamma^\nu_J + \gamma^\nu_{J^\perp} + \gamma^\nu_S = 0. \]

(In some complicated form depending on both \(e\) and \(p_t\), will show simple cancelation explicitly in next example for \(p_T\) spectrum.)
Running Strategy

- **Natural scales:**
  - **hard function**: independent of $\nu$, $\mu_H = \sqrt{s}$
  - **soft function** $(\nu_S, \mu_S) = (\sqrt{s} e, \sqrt{s} e)$
  - **jet functions** $(\nu_J, \mu_J) = (\sqrt{s}, \sqrt{s} e)$

- **2-Parameter RG**
  - UV- and Rapidity-Divergences are independent
  - $\mu$ and $\nu$ RG are independent (commute)
  - $\partial_\mu \partial_\nu F = \partial_\nu \partial_\mu F$, where $F$ can be jet function $J(\nu, \mu)$ or soft function $S(\nu, \mu)$
Running Strategy

- Natural scales:
  - hard function: independent of $\nu$, $\mu_H = \sqrt{s}$
  - soft function $(\nu_S, \mu_S) = (\sqrt{s} e, \sqrt{s} e)$
  - jet functions $(\nu_J, \mu_J) = (\sqrt{s}, \sqrt{s} e)$

- Running
  - In $\mu$:
    Evolve hard function from high scale $\mu_H = \sqrt{s}$ to common low scale $\mu_J = \mu_S = \sqrt{s} e$
  - In $\nu$:
    Evolve soft function from $\nu_S = \sqrt{s} e$ to jet scale $\nu_J = \sqrt{s}$
Solution to the $\mu$-RGE for the hard function

$$H(s, \mu) = U_H(s; \mu_H, \mu)H(s; \mu_H)$$

with

$$U_H(s; \mu_H, \mu) = \left| e^{K_H(\mu_H, \mu)} \left( \frac{-s - i0}{\mu_H^2} \right)^{\eta_H(\mu_H, \mu)} \right|^2$$

Solution to the $\nu$-RGE for the soft function at one-loop

$$S^R(\tau, b_{1t}, b_{2t}, \mu, \nu) = \left( \frac{\mu e^{\gamma_E} (\tau + \sqrt{b_{1t}^2 Q^2 + \tau^2})}{2Q} \right)^{\zeta} \left( \frac{\mu e^{\gamma_E} (\tau + \sqrt{b_{2t}^2 Q^2 + \tau^2})}{2Q} \right)^{\zeta} \times S^R(\tau, b_{1t}, b_{2t}, \mu, \nu_S)$$

in which $\zeta = -2 \frac{\alpha_s}{\pi} C_F \ln \frac{\mu^2}{\nu^2}$. 
NLO singular cross-section

\[
\frac{1}{\sigma_0} \frac{d\sigma}{de} = \frac{1}{e} \frac{\alpha_s(\mu) C_F}{\pi} \left( -3 - 4 \log \frac{e}{2} \right)
\]

Resummed cross-section up to NLL

\[
\frac{1}{\sigma_0} \frac{d\sigma}{de} = H(Q, \mu) \frac{4 e^{2 \gamma_E \zeta}}{\Gamma(-2\zeta)} \frac{1}{eQ} \left( \frac{\mu}{eQ} \right)^{2\zeta} \left( 1 - 2 F_1(1, 1; 1 - \zeta; -1) \right)^2
\]

To compare with standard Total Jet Broadening observable $B_T$, recall

\[
e = 2B_T
\]
Comparison with literature...

QCD calculations with resummation in the literature:

1. Catani, Turnock and Webber (1992) $\Rightarrow$ Correct up to NLL. Agree with our NLL result.

   $\Rightarrow$ Corrections to our previous result in [1] at NLL’ or NNLL.
   $\Rightarrow$ Agree with updated result in this talk

SCET resummation for $B_T$ after our letter...

   $\Rightarrow$ Claim to agree with [2] for all logs at $\alpha_S^2$ order.
   $\Rightarrow$ Factorization broken by regulator
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Comparison with data

\[ \frac{d\sigma}{dB_T} \text{ (nbarn)} \]

\[ Q = 130 \text{ GeV} \]

Data from L3

- NLO sing.
- LL
- NLL

\[ \bullet \] Data from L3

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Other Applications?

Rapidity divergences do not only appear when observing Jet Broadening...

- $p_T$ spectrum for Higgs/Drell-Yan Production
- TMD-PDF, Generalized Parton Distribution
- Electroweak corrections to high energy process at the LHC
- ...

In general, processes or observables involving collinear and soft mode with similar transverse momentum, invariant mass, or off-shellness.
$p_T$ Resummation in SCET

- **Idilbi, Ji, Yuan (2005)**
  - Calculation using SCET, no factorization theorem derived

- **Mantry and Petriello (2009, 2010)**
  - Factorization theorem derived in SCET
  - Keep residual momentum, and thus power suppressed terms for each sector to be well regularized.

- **Becher and Neubert (2010)**
  - Absence of soft function
  - Analytic regulator break factorization
When observing $p_T$ (or related observables)

- When measuring thrust distribution or $|p^s_\perp| \ll |p^n_\perp| \sim |p^{\bar{n}}_\perp|$ and $|\eta^s| \ll |\eta^n| \sim |\eta^{\bar{n}}|$. Ultra-Soft radiation decoupled from the collinear radiation.

- When observing transverse momentum ($p_T$) distribution or jet broadening event shape ($B_T$), $k_{S,\perp} \sim k_{n,\perp} \sim k_{\bar{n},\perp}$, both soft and collinear radiation contribute to the transverse momentum at the same order.

$$k_S \sim (p_T, p_T, p_T)$$

$$J_n : k_n \sim (Q, \sqrt{p_T^2/Q}, p_T)$$

$$J_{\bar{n}} : k_{\bar{n}} \sim (\sqrt{p_T^2/Q}, Q, p_T)$$

$$p_H \sim (p^-_H, p^+_H, p_T)$$
Higgs $\rho_T$ distribution in SCET$_{II}$
with $\eta$-regulator and $\nu$-RG

\[
\frac{d\sigma}{dQ^2 dp_T^2 dy} \propto \int \frac{dx_1}{x_1} \int \frac{dx_2}{x_2} \delta(\vec{p}_T^H + \vec{p}_n, t + \vec{p}_\bar{n}, t + \vec{p}_s, t)\frac{g^\perp_{\alpha\sigma} g^\perp_{\beta\omega}}{x_1 x_2} \\
\times H(m_H^2, \mu)S(\vec{p}_s, t; \mu_s, \mu; \nu_s, \nu) \\
\times \mathcal{F}^{\alpha\beta}_{n; g} (x_1, \vec{p}_n, t; \mu_B, \mu; \nu_B, \nu) \\
\times \mathcal{F}^{\sigma\omega}_{\bar{n}; g} (x_2, \vec{p}_{\bar{n}}, t; \mu_B, \mu; \nu_B, \nu)
\]

\[
f_{g/p}(\frac{\omega_a}{P^-}, \mu) = -\sum_{\text{spins}} \theta(\omega_a)\omega_a \langle p_n | B^{c\mu}_{n\perp}(0) \delta(\frac{\omega_a}{P^-} - \vec{P}_n) B^{c}_{n\perp\mu}(0) | p_n \rangle
\]

\[
\mathcal{F}^{\alpha\beta}_{g} (\frac{\omega_a}{P^-}, \vec{p}_t, \mu) = -\sum_{\text{spins}} \int d^2 \vec{b}_t e^{-i \vec{b}_t \cdot \vec{p}_t} \theta(\omega_a) \langle p_n | B^{c\alpha}_{n\perp}(\vec{b}_t) \delta(\frac{\omega_a}{P^-} - z) B^{c\beta}_{n\perp}(0) | p_n \rangle
\]

\[
= \sum_i \frac{1}{z} \int_{z'}^1 \frac{dz'}{z} \int d^2 \vec{b}_t e^{-i \vec{b}_t \cdot \vec{p}_t} f_i (\frac{\omega_a}{z' P^-}, \vec{b}_t, \mu) f_i (z', \mu)
\]

- divergent/ill-defined integral by pure dim-reg.
Beam Function Calculation with $\eta$-regulator

Total beam function in $n$-direction including real and virtual yields

$$\mathcal{F}_{g\leftarrow g}(z, \vec{p}_t) \propto \Gamma(1 + \epsilon) \frac{\mu^{2\epsilon}}{(p_t^2)(1+\epsilon)} \left[ g_t^{\mu\nu} \frac{\delta(1 - z)}{\eta} \left( \frac{\nu}{\omega_a} \right)^{\eta} + \rho_{gg} \left( \frac{1}{z} \right) g_t^{\mu\nu} \right.$$

$$\left. + \frac{4(1 - z)}{z^2} \left( \frac{p_t^{\mu} p_t^{\nu}}{p_t^2} + \frac{1}{2} g_t^{\mu\nu} \right) + 2\epsilon \left( \frac{p_t^{\mu} p_t^{\nu}}{p_t^2} - \frac{1}{2\epsilon} g_t^{\mu\nu} \right) \right]$$

- splitting function

$$\rho_{gg}(z) = \frac{1 + (1 - z)^4 + z^4}{[1 - z]_+}$$

- $$(\mu_B, \nu_B) = (p_t, \omega)$$
Soft Function Calculation with $\eta$-regulator

\[ S(\vec{p}_t) \propto \Gamma \left( 1 + \epsilon + \frac{\eta}{2} \right) \Gamma \left( \frac{\eta}{2} \right) \left( \frac{\mu^{2\epsilon}}{(p_t^2)^{(1+\epsilon)}} \right) \left( \frac{\nu^\eta}{(p_t^2)^{\eta/2}} \right) \]

\[ (\mu_s, \nu_s) = (p_t, p_t) \]
The rapidity divergences for the jet and soft functions introduce a new set of anomalous dimension \((\gamma_\nu^B, \gamma_\nu^S)\):

\[
\nu \frac{d}{d\nu} S^R(\mu, \nu) = \gamma_\nu^S S^R(\mu, \nu), \quad \nu \frac{d}{d\nu} B_n^R(\mu, \nu) = \gamma_\nu^B B_n^R(\mu, \nu)
\]

Just like the traditional \(\mu\) anomalous dimension:

\[
\mu \frac{d}{d\mu} S^R(\mu, \nu) = \gamma_\mu^S S^R(\mu, \nu), \quad \mu \frac{d}{d\mu} B_n^R(\mu, \nu) = \gamma_\mu^B B_n^R(\mu, \nu)
\]

In impact parameter space

\[
\gamma_\nu^S = 2 \frac{\alpha_s}{\pi} C_A \ln \left( \frac{b^2 \mu^2 e^{2\gamma_E}}{4} \right) \quad \text{and} \quad \gamma_\nu^B = - \frac{\alpha_s}{\pi} C_A \ln \left( \frac{b^2 \mu^2 e^{2\gamma_E}}{4} \right)
\]

\[
\gamma_\mu^S = -2 \frac{\alpha_s}{\pi} C_A \ln \left( \frac{\mu^2}{\nu^2} \right) \quad \text{and} \quad \gamma_\mu^B = - \frac{\alpha_s}{\pi} C_A \ln \left( \frac{\nu^2}{\omega_n^2} \right)
\]
Since the cross-section is invariant under $\mu$ and $\nu$ variation, and that the hard function itself is free from rapidity divergence (and therefore $\gamma_H^\nu = 0$), we must have the relations

$$\gamma_H^\mu + \gamma_B^\mu + \gamma_{B_n}^\mu + \gamma_S^\mu = 0, \text{ and } \gamma_B^\nu + \gamma_{B_n}^\nu + \gamma_S^\nu = 0.$$ 

In impact parameter space

$$\gamma_H^\mu = 2\frac{\alpha_s}{\pi} C_A \ln\left(\frac{\mu^2}{\omega_n \omega_{\bar{n}}}\right)$$

$$= 2\frac{\alpha_s}{\pi} C_A \ln\left(\frac{\mu^2}{\nu^2}\right) + \frac{\alpha_s}{\pi} C_A \ln\left(\frac{\nu^2}{\omega_n^2}\right) + \frac{\alpha_s}{\pi} C_A \ln\left(\frac{\nu^2}{\omega_{\bar{n}}^2}\right)$$

$$0 = 2\frac{\alpha_s}{\pi} C_A \ln\left(\frac{b^2 \mu^2 e^{2\gamma_E}}{4}\right) - 2\frac{\alpha_s}{\pi} C_A \ln\left(\frac{b^2 \mu^2 e^{2\gamma_E}}{4}\right)$$

Recover CSS formula by $\mu$ and $\nu$ RG
Solution to the $\mu$-RGE for the hard function

$$H(s, \mu) = U_H(M_H; \mu_H, \mu) H(M_H; \mu_H)$$

with

$$U_H(M_H; \mu_H, \mu) = \left| e^{K_H(\mu_H, \mu)} \left( \frac{-M_H^2 - i0}{\mu_H^2} \right) \eta_H(\mu_H, \mu) \right|^2$$

Solution to the $\nu$-RGE for the soft function

$$S(\mu, \nu) = V_S(\mu, \frac{\nu}{\nu_S}) \otimes S(\mu, \nu_S)$$

with

$$V_S(p_t; \omega_s, \mu, \nu) = e^{-2\gamma_E \omega_s} \frac{\Gamma(1 - \omega_s)}{\Gamma(1 + \omega_s)} \left[ \frac{\omega_s}{\mu} \left[ \frac{1}{(\frac{p_t}{\mu})^{\frac{1}{1-\omega_s}}} + \delta(p_t) \right] \right]$$

and

$$\omega_s(\mu, \frac{\nu}{\nu_S}) = 2\Gamma_{cusp}[\alpha_s(\mu)] \log \frac{\nu}{\nu_S}.$$ 

Resummed cross-section up to NLL

$$\frac{1}{\sigma_0} \frac{d\hat{\sigma}}{d\rho_T} = U_H(M_H; \mu_H, \mu = p_T) V_S(p_t; \omega_s, \mu = p_T, \nu = M_H)$$
Comparison with literature...

QCD calculations with resummation in the literature:


SCET resummation for $p_T$ distribution

2. Gao, Li, Liu (2005)

3. Idilbi, Ji, Yuan (2005)
   - SCET-like calculation, no factorization theorem derived
   - log hidden in phase space

4. Mantry, Petriello
   - Factorization theorem derived in SCET
   - Keep residual momentum, and thus power suppressed terms for each sector to be well regularized.
   - log hidden in phase space

   - Absence of soft function
   - Analytic regulator break factorization
Conclusion

- When measuring transverse momentum related observables...
  - Soft contributions are important
  - Uncanceled divergences remain in each sector, rapidity divergence.
  - New kind of logarithms to resum, yet related to the cups angle (the high scale).
- There are other cases with rapidity divergence such as, higgs $p_T$ distribution, and electroweak corrections to LHC processes.
- Rapidity RG making use of the $\eta$-regulator provides controllable form to divergences, and a way to resum the log **systematically**.
Jet Broadening Resummation in [3]

Bechera, Bell and Neubert also attempted to resum the rapidity logs for the jet broadening...

\[
\frac{1}{\sigma_0} \frac{d\sigma}{db_T} = H(Q^2, \mu) \frac{e^{-2\gamma_E \eta}}{\Gamma(2\eta)} \frac{1}{b_T} \left( \frac{b_T}{\mu} \right)^{2\eta} l^2(\eta)
\]

where

\[
l(\eta) = \frac{4\eta}{1 + \eta} 2F_1(\eta, 1 + \eta, 2 + \eta, -1)
\]

\[
= 1 + \eta^2 \left[ \frac{\pi^2}{12} - \log^2 2 \right] + O(\eta^3)
\]

and

\[
\eta \equiv \frac{\alpha_S(\mu)}{\pi} C_F \log \frac{Q^2}{\mu^2} \sim \frac{\alpha_S(\mu)}{\pi} C_F \log \frac{1}{b_T^2}
\]

- Not claimed to be correct when we did NLL resum.
- Reproduced by RRG when properly convolving in \( \vec{p}_T \) back
\[ \frac{b_T}{\sigma_0} \frac{d\sigma}{db_T} = \frac{\alpha_s(Q)}{2\pi} A(b_T) + \left( \frac{\alpha_s(Q)}{2\pi} \right)^2 B(b_T) \]

\[ A^{\text{NLL}}(b_T) = C_F (-8L - 6) , \]

\[ B^{\text{NLL}}(b_T) = C_F^2 \left[ 32L^3 + 72L^2 + \left( 92 - \frac{40\pi^2}{3} - 64\ln^2 2 \right) L \right] \]

\[ + C_F C_A \left[ \frac{88}{3} L^2 + \left( \frac{4\pi^2}{3} - \frac{70}{9} \right) L \right] + C_F T_F n_f \left( -\frac{32}{3} L^2 + \frac{8}{9} L \right) \]