Effective Field Theory and Efimov Physics

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Agenda

- Introduction
- Resonant Interactions and Efimov Physics
- Effective Field Theory for Large Scattering Length
- Applications
  - Ultracold atoms
  - Hadronic Molecules
- Summary and Outlook
Effective Field Theory

- Limited resolution in low-energy processes

\[ x + = \ldots + \]

- Typical momentum: \( p = \frac{\hbar}{L} \Rightarrow \text{resolution: } |x - y| \gtrsim L \)

- Short-distance physics not resolved
  \( \rightarrow \) capture in low-energy constants using renormalization
  \( \rightarrow \) include long-range physics explicitly

- Systematic, model independent \( \rightarrow \) universal properties

- Classic example: light-light-scattering (Euler, Heisenberg, 1936)

Simpler theory for \( \omega \ll m_e \):

\[ \mathcal{L}_{QED}[\psi, \bar{\psi}, A_\mu] \rightarrow \mathcal{L}_{\text{eff}}[A_\mu] \]

\[ \Rightarrow \] + \ldots
**Low-Energy Universality**

- **Renormalization Group**: Systems with very different fundamental interactions can behave similarly at low energies
  - Universal properties

- **Nuclear Physics**: Nucleons and pions

- **Atomic Physics**: Born-Oppenheimer plus van der Waals

  ⇒ **At sufficiently low energy**: contact interactions
Resonant Interactions

- Large scattering length: $|a| \gg \ell \sim r_e, l_{vdW}, \ldots$
- Natural expansion parameter: $\ell/|a|, k\ell, \ldots$

\[
a > 0 \implies B_d = \frac{1}{2\mu a^2} + \mathcal{O}(\ell/a)
\]

- Atomic physics:
  - $^4\text{He}$: $a \approx 104 \text{ Å} \gg r_e \approx 7 \text{ Å} \sim l_{vdW} \rightarrow B_d \approx 100 \text{ neV}$
  - Feshbach resonances $\implies$ variable scattering length

- Nuclear physics: $S$-wave $NN$-scattering, halo nuclei,…
  - $^1S_0, ^3S_1$: $|a| \gg r_e \sim 1/m_\pi \rightarrow B_d \approx 2.2 \text{ MeV}$
  - $^6\text{He} \implies P$-wave universality?

- Particle physics:
  - $X(3872)$ as a $D^0\bar{D}^{0*}$ molecule? ($J^{PC} = 1^{++}$)
    \[
    B_X = m_{D^0} + m_{D^{0*}} - m_X = (0.3 \pm 0.4) \text{ MeV}
    \]
Three-body system with large scattering length $a$

Hyperspherical coordinates: $R^2 = (r_{12}^2 + r_{13}^2 + r_{23}^2)/3$

Schrödinger equation simplifies for $|a| \gg R \gg l$:

$$-\frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial R^2} + \frac{s_0^2 + 1/4}{R^2} \right] f(R) = -\frac{\hbar^2 \kappa^2}{m} \frac{1}{E} f(R)$$

Singular Potential: renormalization required

Boundary condition at small $R$: breaks scale invariance

$\Rightarrow$ dependence of observables on 3-body parameter (and $a$)

EFT formulation: boundary condition $\Rightarrow$ 3-body interaction
Two-Body System in EFT

- Effective Lagrangian (Kaplan, 1997; Bedaque, HWH, van Kolck, 1999)

\[ \mathcal{L}_d = \psi^\dagger \left( \frac{1}{2} i \partial_t \mathbf{\nabla}^2 + \frac{\mathbf{\nabla}^2}{2m} \right) \psi + \frac{g_2}{4} d^\dagger d - \frac{g_2}{4} (d^\dagger \psi^2 + (\psi^\dagger)^2 d) - \frac{g_3}{36} d^\dagger d \psi^\dagger \psi + \ldots \]

- Interacting dimeron propagator \( \rightarrow \) sum bubbles

- Two-body amplitude \( T_2(k, k) \):

\[ T_2(k, k) = \propto \left[ \frac{8\pi}{g_2 + 2\Lambda/\pi + ik} \right]^{-1} + \ldots \]

- Matching: \( g_2 \leftarrow a, B_d \)

- RG fixed points of \( g_2(\Lambda) \): \( a = 0 \) and \( a = \infty \)

- Higher order corrections: perturbation theory
Three-Body System in EFT

Three-body equation:

\[ T_3(k, p) = M(k, p) + \frac{4}{\pi} \int_0^\Lambda dq \, q^2 \, M(q, p) D_\alpha(q) \, T_3(k, q) \]

with

\[ M(k, p) = F(k, p) - \frac{g_3}{9g_2^2} \left( \frac{H(\Lambda)}{\Lambda^2} \right) \]

\( g_3 = 0, \, \Lambda \to \infty \to \text{Skorniakov, Ter-Martirosian '57} \)

Three-body recombination:
Renormalization

- Observables are independent of regulator/cutoff $\Lambda$

$\Rightarrow$ Running coupling $H(\Lambda)$

- $H(\Lambda)$ periodic: limit cycle
  \[
  \Lambda \to \Lambda e^{n\pi/s_0} \approx \Lambda(22.7)^n
  \]
  (cf. Wilson, 1971)

- Full scale invariance broken to discrete subgroup

\[
H(\Lambda) = \frac{\cos(s_0 \ln(\Lambda/\Lambda_*) + \arctan(s_0))}{\cos(s_0 \ln(\Lambda/\Lambda_*) - \arctan(s_0))}, \quad s_0 \approx 1.00624
\]

- Limit cycle $\iff$ Discrete scale invariance

- Matching: $\Lambda_* \leftarrow B_t, K_3, \ldots \rightarrow \kappa_*, a_*, a'_*$
Limit Cycle: Efimov Effect

- Universal spectrum of three-body states
  (V. Efimov, Phys. Lett. 33B (1970) 563)

- Discrete scale invariance for fixed angle $\xi$

- Geometrical spectrum für $1/a \to 0$

- Ultracold atoms $\Rightarrow$ variable scattering length

\[ B_3^{(n)}/B_3^{(n+1)} \xrightarrow{1/a \to 0} 515.035... \]
Efimov Effect in Finite Volume

- Modification of spectrum by cubic box \((V = L^3)\)
- Box provides infrared cutoff \(1/L \Rightarrow\) calculable in EFT
- Box breaks rotational invariance \(\Rightarrow\) partial wave mixing
- 3-momenta quantized \(\vec{p} = \vec{n} \left(2\pi/L\right) \Rightarrow\) 3d sum equation

Efimov Effect in Finite Volume

- Higher partial waves:
  - few percent effect from $l = 4$ for $L \gtrsim 2R$
  - very small for threshold state

- Indications for universal scaling of finite volume corrections

  $\Rightarrow \quad L_{10\%} \text{ scales with } 1/\sqrt{B_3} \sim R_3$

- First results for 3 nucleons $\Rightarrow$ Lattice QCD

  $\Rightarrow \quad \Delta B_t/B_t \approx 300\% \quad \text{for} \quad L \approx 3 \text{ fm}$

- Limit cycle in finite volume

  (Kreuzer, HWH, in progress)
Extension to 4-body system in effective QM approach

No four-body parameter at LO  
(Platter, HWH, Meißner, 2004)

3- and 4-body observables are correlated  
(Tjon line, ...)
More on the 4-Body System

- Universal properties of 4-body system with large $a$
  - Bound state spectrum, scattering observables, ...

- “Efimov-plot”: 4-body bound state spectrum as function of $1/a$

$$K = \text{sign}(E) \sqrt{m|E|}$$

$$B_4^{(0)} = 5B_3^{(0)} \quad (1/a \equiv 0)$$

$$B_4^{(1)} = 1.01B_3^{(0)}$$

(Platter, HWH, EPJA 32 (2007) 113)

- Improved theoretical description and signature in Cs loss data
  Ferlaino, Knoop, Berninger, Harm, D’Incao, Nägerl, Grimm, PRL 102 (2009) 140401

- Four-body recombination (Wang, Esry; Mehta et al.)
Efimov Physics in Cold Atoms

- **Velocity distribution** ($T = 400 \text{ nK}, 200 \text{ nK}, 50 \text{ nK})$

(Source: http://jilawww.colorado.edu/bec/)

- Few-body loss rates provide window on Efimov physics
- Variable scattering length via Feshbach resonances
Three-Body Recombination

- Recombination into weakly-bound dimer:
  \[ 3 \text{ atoms} \rightarrow \text{dimer} + \text{atom} \Rightarrow \text{loss of atoms} \]
- Recombination constant:
  \[ \dot{n}_A = -K_3 n_A^3 \]
- Scattering length dependence for \( a > 0 \):
  \[ K_3 \approx 201.3 \sin^2 [s_0 \ln(a\kappa_*) + 1.16] \frac{\hbar a^4}{m}, \quad s_0 \approx 1.00624.. \]
- Modification from deep dimers: Efimov states acquire width
  \[ \Rightarrow \quad \kappa_* \rightarrow \kappa_* \exp(i\eta_* / s_0) \]
- Recombination into deep dimers \( \Rightarrow \) Efimov resonances
- Evidence for Efimov trimers in \(^{133}\text{Cs}, ^6\text{Li}, ^7\text{Li}, ^{39}\text{K}\)
Dimer Relaxation in $^{133}$Cs

- Dimer Relaxation: $a + d \rightarrow a + D$ (energetic)
- Relaxation constant: $\dot{n}_A = \dot{n}_D = -\beta n_A n_D$

- Recent experiment: Knoop et al. (Innsbruck), Nature Physics 5 (2009) 227
- Finite temperature $T \sim T_c$: Bose-Einstein average?

\[ \eta_* = 0.034 \quad \eta_* = 0.06 \quad \eta_* = 0.036 \text{ (BO)} \]

Helfrich, HWH, EPL 86 (2009) 53003
Recent experiment with heteronuclear mixture of \(^{41}\text{K}\) and \(^{87}\text{Rb}\) atoms (Barontini et al. (Florence), Phys. Rev. Lett. \textbf{103} (2009) 043201)

⇒ Connected K-Rb-Rb resonances for \(a > 0\) and \(a < 0\)

Ratio of resonance positions: \(\frac{a_*/|a_-|}{\sqrt{1}}\)

Helfrich, HWH, Petrov, arXiv:1001.4371

\textbf{K-Rb-Rb:} \(\frac{m_1}{m_2} = \frac{41}{87} \approx 0.47\) \(⇒\) \(\exp(\pi/s_0) \approx 128\)

\(a_*/|a_-|:\) \(2.7\) (Exp) \(⇒\) \(0.52\) (Th)
Heteronuclear Efimov Effect

- Identification of atom-dimer resonance in purely atomic sample through rescattering processes? (cf. Barontini et al., 2009)
- Calculate loss rate above threshold: \( E/B_d = -1, -0.95, -0.5, 0 \)

\[ \eta_* = 0.1 \]

- Should not exclude other interpretations of data
- Analytical and numerical results for mass dependence of recombination and relaxation rates (cf. D’Incao, Esry, 2006)
P-Wave Universality?

- Universal properties for resonant $P$-wave interactions?
  $\implies$ not in the usual sense

- Interactions corresponding to $a_1$ and $r_1$ required for consistent renormalization (Bertulani, HWH, van Kolck, 2002)

- Two relevant directions near RG fixed point (Barford, Birse, 2003)

- Causal wave propagation requires $(2L + d \geq 5)$

$$r_{L,d} \leq -2\Gamma(L + \frac{d}{2} - 2)\Gamma(L + \frac{d}{2} - 1)/\pi \times (2\Lambda)^{2L+d-4}$$

- $d = 3, L = 0 \implies r_{0,3} \leq 2/\Lambda$

- $d = 3, L \geq 1 \implies r_{0,3} \leq -|\text{const.}|\Lambda^{2L-1}$

$\Rightarrow$ effective range cannot be tuned to zero
Exotic Charmonium Mesons

- Many new $c\bar{c}$-mesons at B-factories: $X$, $Y$, $Z$
  - Challenge for understanding of QCD
  - Large scattering length physics important
- Example: $X(3872)$ (Belle, CDF, BaBar, D0)

$\begin{align*}
m_X &= (3871.55 \pm 0.20) \text{ MeV} \\
\Gamma &< 2.3 \text{ MeV} \\
J^{PC} &= 1^{++}
\end{align*}$

- No ordinary $c\bar{c}$-state
  - Decays violate isospin
  - Measured mass depends on decay channel
- Nature of $X(3872)$?
  - $D^0 D^{0*}$-molecule? (cf. Tornquist, 1991)
  - Tetraquark
  - Charmonium Hybrid
  - ...
Nature of $X(3872)$

- Nature of $X(3872)$ not finally resolved

- Assumption: $X(3872)$ is weakly-bound $D^0$-$\bar{D}^0*$-molecule

$$\implies |X\rangle = (|D^0 \bar{D}^0*\rangle + |\bar{D}^0 D^0*\rangle)/\sqrt{2}, \quad B_X = (0.26 \pm 0.41) \text{ MeV}$$

- universal properties (cf. Braaten et al., 2003-2008, ...)

  - Explains isospin violation in decays of $X(3872) \Rightarrow$ superposition of $I = 1$ and $I = 0$
  
  - Different masses due to different line shapes in decay channels

- EFT with explicit pions: short distance contributions dominate
  (Fleming, Kusunoki, Mehen, van Kolck, 2007)

  $$\implies \text{EFT for large scattering length is applicable}$$

- Large scattering length determines interaction of $X(3872)$ with $D^0$ and $D^0*$
Interactions of $X(3872)$

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- Efimov effect?
  - Occurs if 2 out of 3 pairs have resonant interactions

- $X(3872)$: only 3 out of 6 pairs have resonant interactions
  - No Efimov effect (Braaten, Kusunoki, 2003)
  - No $X-D^0$- and $X-D^{0*}$-molecules
  - No three-body interaction at leading order
Interactions of $X(3872)$

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- But: parameter-free prediction of $X$-$D^0$-, $X$-$D^{0*}$-scattering

- Low-energy parameters: $B_X = (0.26 \pm 0.41)$ MeV
  ⇒ Scattering length in the $X$ channel: $a = (8.8^{+\infty}_{-3.3})$ fm
Predictions for scattering amplitude/cross section


Three-body scattering lengths

\[ a_{D^0X} = a_{\bar{D}^0X} = -9.7a, \quad \text{and} \quad a_{D^{*0}X} = a_{\bar{D}^{*0}X} = -16.6a \]
Behavior of $X(3872)$ produced in isolation should be distinguishable from its behavior when in the presence of $D^0, D^{*0}, \bar{D}^0, \bar{D}^{*0}$

Rare events in $B\bar{B}$ production ($B \rightarrow X, \bar{B} \rightarrow D, D^*$)

Final state interaction of $D, D^*$ mesons in $B_c$-decays

Example: quark-level $B_c$ decay yielding three charmed/anticharmed quarks in final state

Process may be accessible at the LHC
Summary and Outlook

- Effective field theory for large scattering length
  - Limit cycle in three-body system ⇐ Efimov physics
  - Universal correlations (Phillips, Tjon line, ...)

- Applications in atomic, nuclear, and particle physics
  - Cold atoms close to Feshbach resonance
  - Halo nuclei
  - Scattering properties of the $X(3872)$

- Future directions:
  - Cold atoms: heteronuclear systems, $N \geq 4$, 2d-systems, ...
  - Halo nuclei: reactions, external currents, ...
  - Hadronic molecules: universal properties, three-body molecules?
  - Three-nucleon system on the lattice: finite volume corrections, limit cycle in “deformed” QCD?