Inverse square potential, scale anomaly, and complex extension

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Outline

• Introduction and motivation
• Functional renormalization group and RG flow equation
• Complex extension and RG flows on the Riemann sphere
• Connection to large-flavor QCD
Introduction

• Inverse square potential is classically scale invariant

\[ V(r) = -\frac{\kappa}{r^2} \]

• Classical scattering is well defined if

\[ b > \sqrt{\frac{\kappa}{E}} \quad \text{for} \quad \kappa > 0 \]

otherwise → fall to the center

• QM estimate: particle confined to a small ball of radius \( r_0 \)

\[ E \approx \frac{1}{r_0^2} - \frac{\kappa}{r_0^2} \]

• It is a border between regular and singular potentials
**Introduction**

- In QM a critical $\kappa_{cr} > 0$ exists. For $\kappa > \kappa_{cr}$ the Hamiltonian is unbounded from below $\rightarrow$ problems

- Effective 1D Schrödinger equation

  \[
  \left[ -\frac{d^2}{dr^2} - \frac{\kappa}{r^2} \right] \psi(r) = E \psi(r) \quad E = -\sigma^2 < 0
  \]

- Solution, which is well-behaved as $r \rightarrow \infty$

  \[
  \psi(r) = \sqrt{r} K_\nu(\sigma r) \quad \nu = \sqrt{1/4 - \kappa}
  \]
Introduction

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$$\psi(r) = \sqrt{r} K_\nu(\sigma r) \quad \nu = \sqrt{1/4 - \kappa}$$

• Infinity of nodes $\rightarrow$ infinity of bound states for $\kappa > \kappa_{cr}$?

• Proper treatment: regularize around the origin and use RG

• A lot of previous RG studies

Beane et al. 01, Bawin&Coon 03, Barford&Birse 03 . . .
Motivation

• Efimov effect for three bosons interacting through a short range attractive potential

At unitarity point $\rightarrow$ effective 1D equation

\[
\left[-\frac{d^2}{dr^2} - \frac{s_0^2 + 1/4}{r^2}\right] \psi(r) = E\psi(r), \quad s_0 \approx 1.0062
\]

Experimentally observed in cold atoms

• A neutral polar molecule interacts with an electron via

\[
V(\vec{r}) \sim \frac{\cos \theta}{r^2}
\]

can be reduced to the isotropic form
Motivation

- Transition from the conformal ($\kappa < \kappa_{cr}$) to the nonconformal ($\kappa > \kappa_{cr}$) regime resembles the BKT phase transition in two dimensions \(^{\text{Kaplan et al. 09}}\)

- Scalar field near the Reissner-Nordström black hole background \(^{\text{Camblong et al. 03}}\)

- AdS/CFT correspondence: scalar field in the anti-de Sitter spacetime $AdS_{d+1}$

\[
\partial_r^2 \phi - \frac{d-1}{r} \partial_r \phi - \frac{m^2}{r^2} \phi - q^2 \phi = 0, \quad q^2 = (q^0)^2 + \vec{q}^2
\]

after change of field $\phi = r^{(d-1)/2} \psi$ one gets

\[
-\partial_r^2 \psi + \frac{m^2 + (d^2 - 1)/4}{r^2} \psi = -q^2 \psi
\]
Functional renormalization group

- Effective average action $\Gamma_k[\phi]$ solves

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{\partial_k R_k}{\Gamma_k^{(2)}[\phi] + R_k}$$

- RG flow in theory space

- Truncation needed to solve the flow equation
Model

• Our model in Euclidean QFT formulation

\[ \Gamma_k[\psi, \psi^*] = \int_Q \psi^*(Q)[i\omega + \vec{q}^2]\psi(Q) - \]
\[ - \kappa \int_{Q_1, \ldots, Q_4} F_d(l) \psi^*(Q_1)\psi(Q_2)\psi^*(Q_3)\psi(Q_4)\delta(-Q_1 + Q_2 - Q_3 + Q_4) \]
\[ - \frac{\lambda}{2} \int_{Q_1, \ldots, Q_4} \psi^*(Q_1)\psi(Q_2)\psi^*(Q_3)\psi(Q_4)\delta(-Q_1 + Q_2 - Q_3 + Q_4), \]

with the Fourier transform of \(1/r^2\) potential in \(d\) dim

\[ F_d(l) = \frac{(4\pi)^{d/2}\Gamma(d/2 - 1)|\vec{l}|^{2-d}}{4} \quad d > 2 \]

• \(\lambda\) is an emergent coupling generated by quantum loops

• We use sharp regulator

\[ R_k(L) = (i\omega + l^2) \left( \frac{1}{\theta(l^2 - k^2)} - 1 \right) \]
Flow equation

- The propagator is not renormalized in the nonrelativistic vacuum
- The long-range potential coupling $\kappa$ is not renormalized
- RG flow for the coupling $\lambda$

\[
\partial_t \lambda_{\psi R} = -\lambda_{\psi R}^2 \left[ -\frac{2\kappa}{d-2} + d - 2 \right] \lambda_{\psi R} - \frac{\kappa^2}{(d-2)^2}
\]

- $\lambda$ is taken to be momentum-independent
- Flow equation for a rescaled dimensionless $\lambda_{\psi R}$
Solution of the flow equation

\[
\frac{d}{dt} \lambda_{\psi R}(t) = \alpha \lambda_{\psi R}(t)^2 + \beta \lambda_{\psi R}(t) + \gamma
\]

- The solution is determined by the sign of the discriminant

\[
D = \beta^2 - 4\alpha\gamma
\]

- \( \kappa < \kappa_{cr} \rightarrow D > 0 \rightarrow \) two fixed points (CFT)
- \( \kappa = \kappa_{cr} \rightarrow D = 0 \rightarrow \) single fixed point \( \rightarrow \kappa_{cr} = \frac{(d-2)^2}{4} \)
- \( \kappa > \kappa_{cr} \rightarrow D < 0 \rightarrow \) no fixed points (limit cycle)
Complex extension– analytical treatment

• For a deeper mathematical understanding we perform a complex extension

\[ \lambda \rightarrow \lambda_1 + i\lambda_2 \]

\[ \partial_t \lambda = \alpha \lambda^2 + \beta \lambda + \gamma \rightarrow \begin{cases} 
\partial_t \lambda_1 = \alpha \lambda_1^2 - \alpha \lambda_2^2 + \beta \lambda_1 + \gamma \\
\partial_t \lambda_2 = 2\alpha \lambda_1 \lambda_2 + \beta \lambda_2 
\end{cases} \]

• The analytical solution is

\[ \lambda(t) = \frac{1}{2\alpha} \left( -\beta - \sqrt{D} \frac{e^{\frac{\sqrt{D}t}{2}} - Ce^{-\frac{\sqrt{D}t}{2}}}{e^{\frac{\sqrt{D}t}{2}} + Ce^{-\frac{\sqrt{D}t}{2}}} \right) \]

• \( C \) determines initial condition for \( \lambda \) in complex plane

• RG trajectories have a constant positive curvature
Complex extension—numerical treatment

- \( D > 0 \): two real fixed points with real eigenvalues

- Divergences are regularized \( \rightarrow \) metastable resonances

- Flow of the imaginary part determines the decay width
Complex extension– numerical treatment

- $D < 0$: two complex fixed points with imaginary eigenvalues

- For $D = 0$ two fixed points merge
  - Intuitive understanding of the transition
Complex extension—physical motivation

• Partial wave expansion for a scattering amplitude

\[
f(p, \theta) = \sum_{l=0}^{l=\infty} (2l + 1) f_l(p) P_l(\cos(\theta))
\]

• For short-range forces s-wave dominates at low energies

\[
f_0 = \frac{1}{g_0(p^2) - ip} \approx \frac{1}{-a^{-1} + \frac{1}{2} r_{eff} p^2 - ip}
\]

• Complex scattering length \( a = \alpha + i\beta \) opens an inelastic scattering channel

• Bound states with finite decay width \( \rightarrow \) resonances
Riemann sphere

- The complex plane $\mathbb{C}$ can be extended by an additional point $\infty$
- The extended complex plane is mapped onto the Riemann sphere via the stereographic projection
Flows on the Riemann sphere: $D > 0$

- Two fixed points on the great real circle
Flows on the Riemann sphere: $D < 0$

- Real flow forms an infinite limit cycle in the complex plane with periodic discontinuities
- On the Riemann sphere the flow periodically traverses the great real circle
Connection to large-flavor QCD

- FRG studies of conformal windows and chiral phase transition in many flavor massless QCD (Gies, Jaeckel 06)

- RG $\beta$-functions of the fermionic self-interactions $\lambda_i$ at a fixed gauge coupling $\alpha$

- $\beta$-functions are similar to our problem

- However, gauge coupling $\alpha$ is running in QCD

- No limit cycles, but chiral symmetry breaking for $\alpha > \alpha_{cr}$
Conclusions

- Nonrelativistic inverse square potential has different physical applications
- It is a paradigm for nonrelativistic conformal symmetry and scale anomaly
- It must be regularized at origin → need for RG
- Complex extension provides a deeper mathematical understanding and is physically motivated
- Geometric description → flows on the Riemann sphere

More can be found in Annals of Physics 325, 491 (2010)
Extra slides
We use quantum field theory methods to investigate nonrelativistic $1/r^2$ problem.

In quantum field theory in Euclidean formulation, we have:

$$e^{W[J]} = \int \mathcal{D}\varphi e^{-S[\varphi]+\int J\varphi}$$

From $W[J]$ we can extract connected correlation functions:

$$\Downarrow$$

scattering amplitudes, bound state energies

However, the functional integral for $W[J]$ is difficult to evaluate.
Functional renormalization group

- Renormalization group idea- introduce regulator

\[ e^{W_k[J]} = \int \mathcal{D}\varphi \exp \left( -S[\varphi] - \frac{1}{2} \int \varphi R_k \varphi + \int J \varphi \right) \]

and study sliding scale \( k \) dependence

- Effective average action \( \Gamma_k[\phi] \) is a Legandre transform of \( W_k[J] \)

- Regulator \( R_k \) introduces scheme dependence in the problem

- For \( k = 0 \) we recover the effective action \( \Gamma[\phi] \Rightarrow 1\text{PI vertices} \Rightarrow \text{correlation functions} \)
Solution of the flow equation

- $D > 0$ and $\lambda_{\psi R}^{IR} < \lambda_{\psi R} < \lambda_{\psi R}^{UV}$: smooth interpolation between two fixed points

$$\lambda_{\psi R}(t) = \frac{-\beta - \sqrt{D} \tanh \left[ \frac{\sqrt{D}}{2} (t + \eta) \right]}{2\alpha}$$

- $D = 0$: logarithmic running $\rightarrow$ Landau pole

$$\lambda_{\psi R}(t) = \lambda_{\psi R}^* - \frac{1}{\alpha t + \eta}, \quad \lambda_{\psi R}^* = -\frac{\beta}{2\alpha}$$

- $D < 0$: periodic infinities $\rightarrow$ geometric bound spectrum

$$\lambda_{\psi R}(t) = \frac{-\beta + \sqrt{-D} \tan \left[ \frac{\sqrt{-D}}{2} (t + \eta) \right]}{2\alpha}$$