Factorization, Evolution and Soft factors

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Outline

- Collinear factorization, PDFs, correlation functions
- Evolution of PDFs and correlation functions
- TMD factorization:
  - TMD distributions, and soft factors
- Evolution of TMD distributions
- Phenomenology
- Summary
Connecting hadrons to partons

- Experiments measure hadrons and leptons, not partons.
- Large momentum transfer – sensitive to partons:
  
  ![Diagram](Diagram)

  Sensitive to partonic dynamics

  +… (Diagrams with more active partons from each hadron!)

  Connection between hadron and parton

- QCD factorization – connecting partons to hadrons:
  
  Hadronic matrix elements of parton fields:

  \[
  \langle p, s | \mathcal{O}(\psi, A^\alpha) | p, s \rangle : \langle p, s | \bar{\psi}(0) \gamma^+ \psi(y) | p, s \rangle, \langle p, s | F^{\alpha}(0) F^{+\beta}(y) | p, s \rangle (-g_{\alpha\beta})
  \]

  Isolate pQCD calculable short-distance partonic dynamics
Factorization and partonic distribution

- Factorization of a physical observable:
  \[ \sigma_{\text{phy}}(Q, 1/R) \sim \hat{\sigma}(Q) \otimes \varphi(1/R) + O(1/QR) \]

- Short-distance – probability
- Power corrections – parton correlations
- Measured
- Long-distance – probability

- Factorization is an approximation!
  **Difficulty:** long-range soft and collinear gluon interaction
  **Complication:** difference in observed physical scales

- Parton distribution:
  \[ \phi(1/R) \propto \langle p, s | \mathcal{O} | p, s \rangle \]

- Explicit form of the operator: \( \mathcal{O} \) arises from the approximation used to derive the factorization
Inclusive DIS

- Color flow, final-state interaction, gauge link:

\[ \sigma_h^{\text{DIS}}(x_B, Q) \approx \sum_f \int dx \phi_f(x, \mu_F^2) \hat{\sigma}_{f/h}(x_B/x, Q^2/\mu_F^2, \alpha_s(\mu)) + \mathcal{O}(1/Q) \]

- Collinear factorization:

\[ q(x, \mu^2) = \int \frac{dy^-}{2\pi} e^{i x p^+ y^-} \langle P | \bar{\psi}(0) \frac{\gamma^+}{2} L_n^\dagger(0, +\infty) L_n(y^-, +\infty) \psi(y^-) | P \rangle \]

- Collinear quark distribution – cut vertex:

\[ \int \frac{d^4 k}{(2\pi)^4} \gamma \cdot n \frac{1}{2P \cdot n} \delta(x - k \cdot n/P \cdot n) \]

Artifact of collinear factorization
Evolution equation

- Evolution equation is a consequence of factorization:

\[ \mu_F^2 \frac{d}{d\mu_F^2} F_2(x_B, Q^2) = 0 \]

- Evolution (differential-integral) equation for PDFs

\[ \sum_f \left[ \mu_F^2 \frac{d}{d\mu_F^2} C_f \left( \frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \right] \otimes \varphi_f(x, \mu_F^2) + \sum_f C_f \left( \frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \otimes \mu_F^2 \frac{d}{d\mu_F^2} \varphi_f(x, \mu_F^2) = 0 \]

- PDFs and coefficient functions share the same logarithms

PDFs: \( \log \left( \frac{\mu_F^2}{\mu_0^2} \right) \) or \( \log \left( \frac{\mu_F^2}{\Lambda_{QCD}^2} \right) \)

Coefficient functions: \( \log \left( \frac{Q^2}{\mu_F^2} \right) \) or \( \log \left( \frac{Q^2}{\mu^2} \right) \)

- DGLAP evolution equation:

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \varphi_i(x, \mu_F^2) = \sum_j P_{i/j} \left( \frac{x}{x'}, \alpha_s \right) \otimes \varphi_j(x', \mu_F^2) \]

Evolution equation is a consequence of factorization!
Evolution kernels of PDFs

- Factorization scale dependence of PDFs:
  \[ \int \frac{d^4 k}{(2\pi)^4} k^\cdot n \frac{1}{2P \cdot n} \delta(x - k \cdot n/P \cdot n) + \text{UVCT}(\mu^2) \]

  from the UVCT – renormalization of the composite operator used to define the PDFs

- Evolution kernels – process independent:
  - extract from the PDFs of a parton state:
    e.g. quark distribution of a quark state
    \[ \mu^2 \frac{\partial}{\partial \mu^2} \phi_{q/q}^{(1)}(x, \mu^2) = P_{qq}^{(1)}(x) \otimes \phi_{q/q}^{(0)}(x, \mu^2) \]
  - Extract from the partonic cross section:
    \[ Q^2 \frac{\partial}{\partial Q^2} \sigma_q^{(1)}(x, Q^2) = P_{qq}^{(1)}(x) \otimes \sigma_q^{(0)}(x, Q^2) \]
  - OPE and renormalization of composite operators:
Almost all existing calculations of SSA are at LO:
- Strong dependence on renormalization and factorization scales
- Artifact of the lowest order calculation

Improve QCD predictions – needed:
- Complete set of twist-3 correlation functions relevant to SSA
- LO evolution for the universal twist-3 correlation functions
- NLO partonic hard parts for various observables
- NLO evolution for the correlation functions, ...

Current status:
- Two sets of twist-3 correlation functions
- LO evolution kernel for $T_{q,F}(x,x)$ and $T_{G,F}^{(f,d)}(x,x)$
- NLO hard part for SSA of $p_T$ weighted Drell-Yan

Kang, Qiu, 2009
Braun, et al 2009
Vogelsang, Yuan, 2009
**Twist-3 distributions relevant to SSA**

- **Two-sets Twist-3 correlation functions:**
  - No probability interpretation!
  - \( \widetilde{T}_{q,F} = \int \frac{dy_1^- dy_2^-}{(2\pi)^2} e^{i x_1 P^+ y_1^-} e^{i x_2 P^+ y_2^-} \langle P, s_T | \overline{\psi}_q(0) \frac{\gamma^+}{2} \left[ \epsilon^{s_T \sigma \eta \bar{n}} F_\sigma^+(y_2^-) \right] \psi_q(y_1^-) | P, s_T \rangle \)

\[ \begin{align*}
\widetilde{T}_{G,F} &= \int \frac{dy_1^- dy_2^-}{(2\pi)^2} e^{i x_1 P^+ y_1^-} e^{i x_2 P^+ y_2^-} \frac{1}{P^+} \langle P, s_T | F^{+\rho}(0) \left[ \epsilon^{s_T \sigma \eta \bar{n}} F_\sigma^+(y_2^-) \right] F^{+\lambda}(y_1^-) | P, s_T \rangle (-g_{\rho\lambda}) \\
\widetilde{T}_{\Delta q,F} &= \int \frac{dy_1^- dy_2^-}{(2\pi)^2} e^{i x_1 P^+ y_1^-} e^{i x_2 P^+ y_2^-} \langle P, s_T | \overline{\psi}_q(0) \frac{\gamma^+ \gamma^5}{2} \left[ i s_T^\sigma F_\sigma^+(y_2^-) \right] \psi_q(y_1^-) | P, s_T \rangle \\
\widetilde{T}_{\Delta G,F} &= \int \frac{dy_1^- dy_2^-}{(2\pi)^2} e^{i x_1 P^+ y_1^-} e^{i x_2 P^+ y_2^-} \frac{1}{P^+} \langle P, s_T | F^{+\rho}(0) \left[ i s_T^\sigma F_\sigma^+(y_2^-) \right] F^{+\lambda}(y_1^-) | P, s_T \rangle (i\epsilon_{\perp\rho\lambda})
\end{align*} \]

- **Twist-2 distributions:**
  - **Unpolarized PDFs:**
    - \( q(x) \propto \langle P | \overline{\psi}_q(0) \frac{\gamma^+}{2} \psi_q(y) | P \rangle \)
    - \( G(x) \propto \langle P | F^{+\mu}(0) F^{+\nu}(y) | P \rangle (-g_{\mu\nu}) \)
  - **Polarized PDFs:**
    - \( \Delta q(x) \propto \langle P, S_{\parallel} | \overline{\psi}_q(0) \frac{\gamma^+ \gamma^5}{2} \psi_q(y) | P, S_{\parallel} \rangle \)
    - \( \Delta G(x) \propto \langle P, S_{\parallel} | F^{+\mu}(0) F^{+\nu}(y) | P, S_{\parallel} \rangle (i\epsilon_{\perp\mu\nu}) \)

- **Evolution equations:**
  See Kang, Qiu, 2009, and Braun, et al 2009
Evolution equations and evolution kernels

- **Evolution equation is a consequence of factorization:**
  
  **Factorization:**
  \[
  \Delta \sigma(Q, s_T) = (1/Q)H_1(Q/\mu_F, \alpha_s) \otimes f_2(\mu_F) \otimes f_3(\mu_F)
  \]

  **DGLAP for \( f_2 \):**
  \[
  \frac{\partial}{\partial \ln(\mu_F)} f_2(\mu_F) = P_2 \otimes f_2(\mu_F).
  \]

  **Evolution for \( f_3 \):**
  \[
  \frac{\partial}{\partial \ln(\mu_F)} f_3 = \left( \frac{\partial}{\partial \ln(\mu_F)} H_1^{(1)} - P_2^{(1)} \right) \otimes f_3.
  \]

- **Evolution kernel is process independent:**
  
  - Calculate directly from the variation of process independent twist-3 distributions  
    Kang, Qiu, 2009
    Yuan, Zhou, 2009
  
  - Extract from the scale dependence of the NLO hard part of any physical process  
    Vogelsang, Yuan, 2009
  
  - Calculate UV renormalization of the operators  
    Braun et al, 2009
Evolution equations – I

- **Feynman diagram representation of twist-3 distributions:**

  Different twist-3 distributions \(\iff\) diagrams with different cut vertices

- **Collinear factorization of twist-3 distributions:**

- **Cut vertex and projection operator in LC gauge:**

\[
\mathcal{V}_{q,F}^{LC} = \frac{\gamma^+}{2P^+} \delta(x - \frac{k^+}{P^+})x_2\delta\left(x_2 - \frac{k_2^+}{P^+}\right)(i\epsilon_{\sigma\mu}^{s\tau\sigma n\bar{n}})[-g_{\sigma\mu}]C_q
\]

\[
P^{(LC)}_{q,F} = \frac{1}{2} \gamma \cdot P\left(\frac{-1}{\xi_2}\right)(i\epsilon_{\sigma\mu}^{s\tau\rho n\bar{n}})\tilde{C}_q
\]
Closed set of evolution equations (spin-dependent):

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{q,F}(x, x + x_2, \mu_F, s_T) = \int d\xi d\xi_2 [\tilde{T}_{q,F}(\xi, \xi + \xi_2, \mu_F, s_T)K_{qq}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) \]

\[ + \tilde{T}_{\Delta q,F}(\xi, \xi + \xi_2, \mu_F, s_T)K_{\Delta q}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) ] \]

\[ + \sum_{i=f,d} \int d\xi d\xi_2 [\tilde{T}_{G,F}(\xi, \xi + \xi_2, \mu_F, s_T)K^{(i)}_{qq}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) \]

\[ + \tilde{T}_{\Delta G,F}(\xi, \xi + \xi_2, \mu_F, s_T)K^{(i)}_{\Delta g}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) ] \]

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}^{(i)}_{G,F}(x, x + x_2, \mu_F, s_T) = \sum_{j=f,d} \int d\xi d\xi_2 [\tilde{T}^{(j)}_{G,F}(\xi, \xi + \xi_2, \mu_F, s_T)K^{(j)}_{gg}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) \]

\[ + \tilde{T}_{\Delta G,F}(\xi, \xi + \xi_2, \mu_F, s_T)K^{(j)}_{\Delta g}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) ] \]

\[ + \sum_q \int d\xi d\xi_2 [\tilde{T}_{q,F}(\xi, \xi + \xi_2, \mu_F, s_T)K^{(i)}_{gq}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) \]

\[ + \tilde{T}_{\Delta q,F}(\xi, \xi + \xi_2, \mu_F, s_T)K^{(i)}_{\Delta g}(\xi, \xi + \xi_2, x, x + x_2, \alpha_s) ] \]

Plus two more equations for:

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{\Delta q,F}(x, x + x_2, \mu_F, s_T) \quad \text{and} \quad \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{\Delta G,F}(x, x + x_2, \mu_F, s_T) \]
Distributions relevant to SSA:

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} T_{q,F}(x, x + x_2, \mu_F) = \frac{1}{2} \left[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{q,F}(x, x + x_2, \mu_F, s_T) + \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{q,F}(x + x_2, x, \mu_F, s_T) \right], \]

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} T_{G,F}^{(i)}(x, x + x_2, \mu_F) = \frac{1}{2} \left[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{G,F}^{(i)}(x, x + x_2, \mu_F, s_T) + \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{G,F}^{(i)}(x + x_2, x, \mu_F, s_T) \right], \]

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} T_{\Delta,q,F}(x, x + x_2, \mu_F) = \frac{1}{2} \left[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{\Delta,q,F}(x, x + x_2, \mu_F, s_T) - \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{\Delta,q,F}(x + x_2, x, \mu_F, s_T) \right], \]

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} T_{\Delta,G,F}^{(i)}(x, x + x_2, \mu_F) = \frac{1}{2} \left[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{\Delta,G,F}^{(i)}(x, x + x_2, \mu_F, s_T) - \mu_F^2 \frac{\partial}{\partial \mu_F^2} \tilde{T}_{\Delta,G,F}^{(i)}(x + x_2, x, \mu_F, s_T) \right]. \]

Important symmetry property:

\[ T_{\Delta,q,F}(x, x, \mu_F) = \int dx_2 [2 \pi \delta(x_2)] T_{\Delta,q,F}(x, x + x_2, \mu_F) = 0, \]

\[ T_{\Delta,G,F}^{(f,d)}(x, x, \mu_F) = \int dx_2 [2 \pi \delta(x_2)] \left( \frac{1}{x} \right) T_{\Delta,G,F}^{(f,d)}(x, x + x_2, \mu_F) = 0. \]

These two correlation functions do not give the gluonic pole contribution directly.
Evolution kernels

Feynman diagrams:

LO for flavor non-singlet channel:
Quark:

\[
\frac{\partial T_{q,F}(x, x, \mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s}{2\pi} \int_{x}^{1} \frac{d\xi}{\xi} \left\{ P_{qq}(z) T_{q,F}(\xi, x, \mu_F) + \frac{C_A}{2} \left[ 1 + \frac{z^2}{1-z} \right] \left[ T_{q,F}(\xi, x, \mu_F) - T_{q,F}(\xi, x, \mu_F) \right] + \frac{C_A}{2} \left[ T_{q,F}(x, x, \mu_F) \right] \right\}
\]

Antiquark:

\[
\frac{\partial T_{\bar{q},F}(x, x, \mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s}{2\pi} \int_{x}^{1} \frac{d\xi}{\xi} \left\{ P_{q\bar{q}}(z) T_{\bar{q},F}(\xi, x, \mu_F) + \frac{C_A}{2} \left[ 1 + \frac{z^2}{1-z} \right] \left[ T_{\bar{q},F}(\xi, x, \mu_F) - T_{\bar{q},F}(\xi, x, \mu_F) \right] + \frac{C_A}{2} \left[ T_{\bar{q},F}(x, x, \mu_F) \right] \right\}
\]

- All kernels are infrared safe
- Diagonal contribution is the same as that of DGLAP
- Quark and antiquark evolve differently – caused by tri-gluon
Leading order evolution equations – II

Gluons:

\[
\frac{\partial T_{G,F}^{(d)}(x, x, \mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \left\{ P_{gg}(z) T_{G,F}^{(d)}(\xi, \xi, \mu_F) \right. \\
+ \frac{C_A}{2} \left[ 2 \left( \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right) \left[ T_{G,F}^{(d)}(\xi, x, \mu_F) - T_{G,F}^{(d)}(\xi, \xi, \mu_F) \right] \\
+ 2 \left( 1 - \frac{1-z}{2z} - z(1-z) \right) T_{G,F}^{(d)}(\xi, x, \mu_F) + (1+z) T_{\Delta G,F}^{(d)}(x, \xi, \mu_F) \right] \\
+ P_{gg}(z) \left( \frac{N_c^2 - 4}{N_c^2 - 1} \right) \sum_q \left[ T_{q,F}(\xi, \xi, \mu_F) + T_{\bar{q},F}(\xi, \xi, \mu_F) \right] \right\}
\]

Similar expression for \( T_{G,F}^{(f)}(x, x, \mu_F) \):

- Kernels are also infrared safe
- Diagonal contribution is the same as that of DGLAP
- Two tri-gluon distributions evolve slightly different
- \( T_{G,F}^{(d)} \) has no connection to TMD distribution
- Evolution can generate \( T_{G,F}^{(d)} \) as long as \( \sum_q [T_{q,F} + T_{\bar{q},F}] \neq 0 \)
Evolution equations for diagonal correlation functions are not closed!

Model for the off-diagonal correlation functions:

For the symmetric correlation functions:

\[
T_{q,F}(x_1, x_2, \mu_F) = \frac{1}{2} \left[ T_{q,F}(x_1, x_1, \mu_F) + T_{q,F}(x_2, x_2, \mu_F) \right] e^{-[(x_1-x_2)^2/2\sigma^2]},
\]

\[
T^{(f,d)}_{G,F}(x_1, x_2, \mu_F) = \frac{1}{2} \left[ T^{(f,d)}_{G,F}(x_1, x_1, \mu_F) + T^{(f,d)}_{G,F}(x_2, x_2, \mu_F) \right] e^{-[(x_1-x_2)^2/2\sigma^2]},
\]

\[
T^{(f,d)}_{G,F}(x_1, x_2, \mu_F) = \frac{1}{2} \left[ T^{(f,d)}_{G,F}(x_1, x_1, \mu_F) + \frac{x_2}{x_1} T^{(f,d)}_{G,F}(x_2, x_2, \mu_F) \right] e^{-[(x_1-x_2)^2/2\sigma^2]}.
\]
Scale dependence of twist-3 correlations

- Follow DGLAP at large $x$
- Large deviation at low $x$ (stronger correlation)

Kang, Qiu, PRD, 2009
TMD factorization

- Need processes with two observed momentum scales:

\[ Q_1 \gg Q_2 \]

necessary for pQCD factorization to have a chance sensitive to parton’s transverse motion

- Example – semi-inclusive DIS:

  - Both \( p \) and \( p' \) are observed
  - \( p'_T \) probes the parton’s \( k_T \)
  - Effect of \( k_T \) is not suppressed by \( Q \)

- Limited processes with valid TMD factorization

  Collins, Qiu, 2007, Vogelsang, Yuan, 2007
  Collins, 2007, Rogers, Mulder, 2010

- TMD distribution is more fundamental, more information

  Semi-inclusive DIS is an ideal place to probe the TMD distributions
  As well as generalized parton distributions (GPD’s)
TMD factorization in QCD – I

- SIDIS as an example – Parton model – LO QCD:
  - Parton distribution function in a hadron
  - Parton-to-hadron fragmentation function

- QCD interaction and leading power regions:
  - Collinear regions:
    \[ k \parallel P_A, \quad k \parallel P_B \]
  - Soft regions
    \[ k^\mu \rightarrow 0 \]
Leading pinch surface:
- hard region
- collinear to P region
- collinear to P' region
- soft region

Factorization:
- TMD parton distribution
- TMD fragmentation
- Soft factors
Factorization formalism:

\[ W^{\mu\nu} \propto \sum_{f} \int d^2k_aT d^2k_bT S(q_{hT} - k_{aT} - k_{bT}) \text{Tr} \left[ \mathcal{P}_{A} J_{f/A}(x, k_{aT}) \bar{P}_{A} H^{\nu}_{f}(Q) \mathcal{P}_{B} J_{f\rightarrow B}(z, k_{bT}) \bar{P}_{B} \bar{H}^{\mu}_{f}(Q) \right] \]

The soft factors:

\[ S = Z_s S^{(0)} \]

\[ S^{(0)}(k_{sT}) = \frac{1}{N_c} \int \frac{dk_s^+ dk_s^-}{(2\pi)^{4-2\epsilon}} \]

The soft factors in collinear factorization:

\[ S^{(0)}(k_{sT}) \rightarrow S^{(0)}(k_{sT} = 0) \rightarrow 1 \]
TMD distributions – commonly used version

- **TMD parton distribution:**

  \[ \tilde{\phi}_{f/A}^{(0)}(x, k_{aT}) = \text{Tr}_{\text{color}} \text{Tr}_{\text{Dirac}} \frac{\gamma^+}{2} \int \frac{dk_a^-}{2\pi} \]

- **TMD fragmentation function:**

  \[ \tilde{D}_{f \rightarrow B}^{(0)}(z, k_{bT}) = \frac{\text{Tr}_{\text{color}} \text{Tr}_{\text{Dirac}}}{N_c} \frac{\gamma^+}{4} \frac{d}{dz} \int \frac{dk_b^-}{2\pi} \]

- **Note:**

  Without integration over \( k_T \), TMD distributions do not have the UV divergence from \( dk_T \), the one responsible for the DGLAP.
Going to the b-space

- **Factorization in b-space:**

  \[
  \tilde{D}_{1, h/f}(z, b_T) = \int d^{2-2\epsilon}k_T e^{ik_T \cdot b_T} D_{1, h/f}(z, zk_T).
  \]

  \[
  W^{\mu\nu} \propto \sum_f W^{\mu\nu}_{(0)f} \int \frac{d^{2-2\epsilon}b_T}{(2\pi)^{2-\epsilon}} e^{-i\vec{q}_T \cdot \vec{b}_T} \tilde{S}(b_T) \tilde{\phi}_{f/A}(x, b_T) \tilde{D}_{f \rightarrow D}(z, b_T)
  \]

- **Universality of the soft factor:**

  \[
  \tilde{S}_0(b_T; y_A, y_B) = \frac{1}{N_c} \langle 0 | W(b_T/2; \infty, n_B)^\dagger_{ca} W(b_T/2; \infty, n_A)^{ab} : W(-b_T/2; \infty, n_B)_{bc} W(-b_T/2; \infty, n_A)^{db} \rangle
  \]

  **Wilson line:**

  \[
  W(x; \infty; n)_{ab} = P \left\{ e^{-ig_0 \int_0^\infty d\lambda \cdot \tilde{A}_0(\alpha)(x+\lambda n)t_\alpha} \right\}_{ab}
  \]

- **Evolution of TMD distributions:**

  from the wave function renormalization and the renormalization of the soft factors
Modified TMD distributions

- Modified TMD distribution – “remove” the soft factor:

\[
\tilde{D}_{1; H_A/f}(z_A, b_T; \zeta_A; \mu) = \lim_{y_A \to +\infty} \tilde{D}_{1; H_A/f}^{\text{unsub}}(z_A, b_T; y_{P_A} - y_B) \left( \frac{\tilde{S}(0)(b_T; y_A, y_n)}{\tilde{S}(0)(b_T; y_A, y_B) \tilde{S}(0)(b_T; y_n, y_B)} \right) \times \text{UV renormalization factor}
\]

- The TMD distribution:

\[
\tilde{f}_{f/H_A}(x, b_T; \zeta_A; \mu) = \lim_{y_A \to +\infty} \lim_{y_B \to -\infty} \tilde{f}_{f/H_A}^{\text{unsub}}(x, b_T; y_{P_A} - y_B) \left( \frac{\tilde{S}(0)(b_T, y_A, y_n)}{\tilde{S}(0)(b_T, y_A, y_B) \tilde{S}(0)(b_T, y_n, y_B)} \right) \times \text{UV renormalization factor}
\]
Evolution of TMD’s:

$$\frac{\partial \ln \tilde{D}_{1,H_A/f}(z_A, b_T; \zeta_A, \ldots)}{\partial \ln \sqrt{\zeta_A}} = \tilde{K}(b_T; \mu)$$

$$\tilde{K}(b_T; \ldots) = \frac{\partial}{\partial y_n} \left[ \frac{1}{2} \ln \tilde{S}(b_T; y_n, -\infty) - \frac{1}{2} \ln \tilde{S}(b_T; +\infty, y_n) \right]$$

$$= \frac{1}{2 \tilde{S}(b_T; y_n, -\infty)} \frac{\partial \tilde{S}(b_T; y_n, -\infty)}{\partial y_n} - \frac{1}{2 \tilde{S}(b_T; +\infty, y_n)} \frac{\partial \tilde{S}(b_T; +\infty, y_n)}{\partial y_n}$$

$$\frac{d \ln \tilde{D}_{1,H_A/f}(z_A, b_T; \zeta_A, \ldots)}{d \ln \mu} = \gamma_D(g; \zeta_A/\mu^2)$$

Evolved TMD’s:

$$\tilde{D}_{1,H_A/f}(z_A, b_T; \zeta_A; \mu) =$$

$$= \tilde{D}_{1,H_A/f}(z_A, b_T; m_A^2/z_A^2; \mu_0) \exp \left\{ \frac{\ln \sqrt{\zeta_A z_A}}{m_A} \tilde{K}(b_T; \mu_0) + \right\}$$

$$+ \int_{\mu_0}^{\mu} \frac{d \mu'}{\mu'} \left[ \gamma_D\left(g(\mu'); 1\right) - \ln \frac{\sqrt{\zeta_A}}{\mu'} \gamma_K\left(\gamma_D\left(g(\mu')\right)\right) \right].$$
CSS b-space resummation formalism

- **Leading order $K_T$-factorized cross section:**

\[
\frac{d\sigma_{AB}}{dQ^2 dQ_T^2} = \sum_f \int d\xi_a d\xi_b \int \frac{d^2k_{At} d^2k_{Bt} d^2k_{s,T}}{(2\pi)^6} \times P_{f/A}(\xi_a, k_{At}) P_{\bar{f}/B}(\xi_b, k_{Bt}) H_{\bar{f}f}(Q^2) S(k_{s,T}) \\
\times \delta^2(\tilde{Q}_T - \tilde{k}_{At} - \tilde{k}_{Bt} - \tilde{k}_{s,T})
\]

\[
\delta^2(\tilde{Q}_T - \prod_i \tilde{k}_{i,T}) = \frac{1}{(2\pi)^2} \int d^2b \ e^{i\tilde{b} \cdot \tilde{Q}_T} \prod_i e^{-i\tilde{b} \cdot \tilde{k}_{i,T}}
\]

\[
\frac{d\sigma_{AB}}{dQ^2 dQ_T^2} \approx \frac{1}{(2\pi)^2} \int d^2b \ e^{ib \cdot \tilde{Q}_T} \tilde{W}_{AB}(b, Q) + Y_{AB}(Q^2, Q_T^2)
\]

No large log’s

resummed

\[
= \frac{1}{(2\pi)} \int_{0}^{\infty} db \ J_0(bQ_T) b\tilde{W}_{AB}(b, Q) + \left[ \frac{d\sigma_{AB}^{(Pert)}}{dQ^2 dQ_T^2} - \frac{d\sigma_{AB}^{(Asym)}}{dQ^2 dQ_T^2} \right]
\]

The $Q_T$-distribution is determined by the b-space function: $b\tilde{W}_{AB}(b, Q)$
The b-space distribution: \[ \tilde{W}_{AB} (b, Q) = \sum_{i,j} \tilde{W}_{ij} (b, Q) \hat{\sigma}_{ij} (Q) \]

- The \( \tilde{W}_{ij} (b, Q) \) obeys the evolution equation
  \[ \frac{\partial}{\partial \ln Q^2} \tilde{W}_{ij} (b, Q) = [K(b\mu, \alpha_s) + G(Q/\mu, \alpha_s)] \tilde{W}_{ij} (b, Q) \]  
  \( (1) \)

- Evolution kernels satisfy RG equations
  \[ \frac{\partial}{\partial \ln \mu^2} K(b\mu, \alpha_s) = -\frac{1}{2} \gamma_K(\alpha_s(\mu)) \]  
  \( (2) \)
  \[ \frac{\partial}{\partial \ln \mu^2} G(Q/\mu, \alpha_s) = \frac{1}{2} \gamma_K(\alpha_s(\mu)) \]  
  \( (3) \)

- CSS Resummation of the large logarithms
  \[ \int \ln \mu^2 \text{ in Eq.}(2) \text{ from } \ln \frac{c^2}{b^2} \text{ to } \ln \mu^2 \]
  \[ \int \ln \mu^2 \text{ in Eq.}(3) \text{ from } \ln Q^2 \text{ to } \ln \mu^2 \]
  \[ \int \ln Q^2 \text{ in Eq.}(1) \text{ from } \ln \frac{c^2}{b^2} \text{ to } \ln Q^2 \]

  \[ c = 2e^{-\gamma_E} \sim 1 \]
Calculation in the b-space

- **Resummed x-section:**

\[
\frac{d\sigma^{(\text{resum})}_{A \rightarrow h}}{dx_B dQ^2 dzdq_{T}^2} \propto \int \frac{d^2 b}{(2\pi)^2} e^{i \vec{q}_T \cdot \vec{b}} W(b, x, z, Q)
\]

\[
W = \begin{cases} 
W_{\text{pert}}^{b}(b, x, z, Q) & b \leq b_{\text{max}} \\
W_{\text{pert}}^{b}(b_{\text{max}}, x, z, Q) F_{NP}^{(b, Q; b_{\text{max}})} & b > b_{\text{max}}
\end{cases}
\]

\[
W_{\text{pert}}^{b}(b, x, z, Q) = \sum_{j} e_{j}^{2} \left[ f_{a/A} \otimes C_{a ightarrow j}^{\text{in}} \right] \left[ C_{j \rightarrow c}^{\text{out}} \otimes D_{b ightarrow h} \right] \times e^{-S(b, Q)}
\]

- **Features:**

  - Sudakov form factor \( b_{sp} \propto \left( \frac{\Lambda_{QCD}}{Q} \right)^{\lambda}, \lambda \approx 0.5 \)
  
  - evolution of \( f_{a/A} \) and \( D_{c \rightarrow h} \) also moves \( b_{sp} \)

  smaller \( \xi \Rightarrow \mu \frac{\partial}{\partial \mu} f_{a/A}(\xi) > 0 \Rightarrow \text{lower } b_{sp} \)
Resummed $p_T$ distribution

Remove the divergence:

- Features:
  - (I): dominated by intrinsic $k_T$ (Gaussian type)
  - (II): pQCD soft-gluon resummation ($q_T \leq Q$)
  - (III): pQCD fixed order calculation ($q_T \sim Q$)
  - relative size of three regions depend on $Q^2$ and $S$
  - large $Q^2$ and large $S' \Rightarrow$ smaller region (I)
  - smaller $Q^2 \rightarrow$ smaller logs $\rightarrow$ smaller region (II)
Works for heavy boson production

- **Upsilon at Tevatron:**
  - Works better for $W/Z$, also works for Drell-Yan, ...
  - Dominated by perturbative small-$b$ contribution in its Fourier conjugate space

- Works better for $W/Z$, also works for Drell-Yan, ...

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Berger, Qiu, Wang, 2005

Qiu, Zhang, 2001

Brock et al, …
Low energy data:

Comparison with Fermilab Drell-Yan data

\[ E_{288} \text{ data} \]
\[ P_{\text{beam}} = 400 \text{ GeV} \]

Much larger power corrections
QCD factorization for SIDIS

- **Factorization:**

- **Low** $P_{hT}$ – TMD factorization:

\[
\sigma_{\text{SIDIS}}(Q, P_{h\perp}, x_B, z_h) = \hat{H}(Q) \otimes \Phi_f \otimes D_{f \rightarrow h} \otimes S + \mathcal{O}\left(\frac{P_{h\perp}}{Q}\right)
\]

- **High** $P_{hT}$ – Collinear factorization:

\[
\sigma_{\text{SIDIS}}(Q, P_{h\perp}, x_B, z_h) = \hat{H}(Q, P_{h\perp}, \alpha_s) \otimes \phi_f \otimes D_{f \rightarrow h} + \mathcal{O}\left(\frac{1}{P_{h\perp}}, \frac{1}{Q}\right)
\]

- **$P_{hT}$ Integrated - Collinear factorization:**

\[
\sigma_{\text{SIDIS}}(Q, x_B, z_h) = \hat{H}(Q, \alpha_s) \otimes \phi_f \otimes D_{f \rightarrow h} + \mathcal{O}\left(\frac{1}{Q}\right)
\]

Collins, Soper
Ji, Ma, Yuan
Also works for HERA data

\[
\sum \int dz \frac{d\sigma_{A \rightarrow h}}{dx_B dQ^2 dz dq^2_T}
\]


It will be interested to see if it works for Jlab energies and future EIC hadron’s transverse momentum distributions
Parton distributions and correlation functions are NOT direct physical observables – could be defined differently

Right definition arises from the approximation used in deriving the factorization!

The evolution equation of the parton distributions and correlation functions is the consequence of the factorization.

The soft factors are universal and could be absorbed into the definition of TMD distributions

Thank you!
What the twist-3 distribution can tell us?

- The operator in Red – a classical Abelian case:

- Change of transverse momentum:

\[ \frac{d}{dt} p'_2 = e (\vec{v}' \times \vec{B})_2 = -ev_3 B_1 = ev_3 F_{23} \]

- In the c.m. frame:

\[ (m, \vec{0}) \rightarrow (1, 0, 0_T), \quad (1, -\hat{z}) \rightarrow (0, 1, 0_T) \]

\[ \implies \frac{d}{dt} p'_2 = e \epsilon^{s_T \sigma n \bar{n}} F_{\sigma}^+ \]

- The total change:

\[ \Delta p'_2 = e \int dy^- \epsilon^{s_T \sigma n \bar{n}} F_{\sigma}^+ (y^-) \]

Net quark transverse momentum imbalance caused by color Lorentz force inside a transversely polarized proton