Small $x$ in ep and eA, theory overview.

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Abstract

Big topic, impossible to give overview. This talk contains personal views on a very simple level, hopefully useful for EIC propaganda purposes and pointing out some open questions.
Outline

- Generalities about small $x$
- Phenomenology machinery
- Evolution equations
- Problems with impact parameter
Nuclear degrees of freedom at small $x$

- At large $x$: large $p^+$, short wavelength (in $x^-$)
Nuclear degrees of freedom at small $x$

- At large $x$: large $p^+$, short wavelength (in $x^-$)
- Decrease $x$: coherently probe larger area
Nuclear degrees of freedom at small $x$

- At large $x$: large $p^+$, short wavelength (in $x^-$)
- Decrease $x$: coherently probe larger area
- Small $x \ll \frac{A^{-1/3}}{m_N R_p}$: coherently probe whole nucleus
Saturation is unitarization of small $x$ DIS

Target rest frame
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Target rest frame

Electron emits virtual photon $\gamma^*$ (this is QED)
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$\gamma^*$ fluctuates into $q\bar{q}$ dipole (still QED)
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Target rest frame

$\gamma^* \text{ fluctuates into } q\bar{q}$ dipole (still QED)

$\gamma^* \text{ emits virtual photon } \gamma^*$ (this is QED)

Small $r$: color transparency

$$\frac{d\sigma_{\text{dip.}}}{d^2 b} \sim \frac{r^2}{\pi R_A^2} \alpha_s(Q^2) xG(x, Q^2)$$
Saturation is unitarization of small $x$ DIS

Target rest frame

\[
l \sim \frac{q^0}{2Q^2} = \frac{1}{x m_N}
\]

$$r \sim \frac{1}{Q}$$

- Electron emits virtual photon $\gamma^*$ (this is QED)
- $\gamma^*$ fluctuates into $q\bar{q}$ dipole (still QED)

Small $r$: color transparency

\[
\frac{d\sigma_{\text{dip.}}}{d^2 b} \sim \frac{r^2}{\pi R_A^2} \alpha_s(Q^2)xG(x, Q^2)
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$xG$ grows as $x^{-\lambda}$
Saturation is unitarization of small $x$ DIS

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$xG$ grows as $x^{-\lambda}$

$\frac{d\sigma_{\text{dip.}}}{d^2b} \sim \frac{r^2}{\pi R_A^2} \alpha_s(Q^2) xG(x, Q^2)$

eventually probability $> 1$

--- unitarization at $\frac{1}{r^2} \sim Q^2 \lesssim Q_s^2(x) \sim \frac{\alpha_s(Q^2)xG(x, Q^2)}{\pi R_A^2} \sim x^{-\lambda}$. 

$I \sim q^0/2Q^2 = 1/xm_N$

$r \sim 1/Q$
Saturation is unitarization of small $x$ DIS

Target rest frame

$\begin{align*}
I \sim \frac{q^0}{2Q^2} = \frac{1}{xm_N} \\
q^0 \sim q_0^2/Q^2 = 1/Q
\end{align*}$

- Electron emits virtual photon $\gamma^*$ (this is QED)
- $\gamma^*$ fluctuates into $q\bar{q}$ dipole (still QED)

Small $r$: color transparency

$xG$ grows as $x^{-\lambda}$

$\frac{d\sigma_{\text{dip.}}}{d^2b} \sim \frac{r^2}{\pi R_A^2} \alpha_s(Q^2)xG(x, Q^2)$

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--- unitarization at $\frac{1}{r^2} \sim Q^2 \lesssim Q_s^2(x) \sim \frac{\alpha_s(Q^2)xG(x, Q^2)}{\pi R_A^2} \sim x^{-\lambda}$.

At small enough $x$ this is a hard scale and $\alpha_s$ is small.
Saturation in partonic language: recombination

Boosting the target

- Radiate $xG(x, Q^2)$ gluons
Saturation in partonic language: recombination

Boosting the target

- Radiate $xG(x, Q^2)$ gluons
- Size of a gluon $\sim 1/k_T$, probe with $Q^2$ resolves the ones with $k_T^2 \lesssim Q^2$
Saturation in partonic language: recombination

- Radiate $xG(x, Q^2)$ gluons
- Size of a gluon $\sim 1/k_T$, probe with $Q^2$ resolves the ones with $k_T^2 \lesssim Q^2$
- Cascade: exponential growth
  \[ xG(x, Q^2) \sim x^{-\lambda} = e^{\lambda y} \]
Saturation in partonic language: recombination

Boosting the target

- Radiate $xG(x, Q^2)$ gluons
- Size of a gluon $\sim 1/k_T$, probe with $Q^2$ resolves the ones with $k_T^2 \lesssim Q^2$
- Cascade: exponential growth
  \[ xG(x, Q^2) \sim x^{-\lambda} = e^{\lambda y} \]

Gluons recombine when they overlap

\[
\frac{1}{k_T^2} \gtrsim \frac{\pi R_A^2}{\alpha_s xG(x, Q^2)} \quad \iff \quad k_T^2 \lesssim \frac{\alpha_s xG(x, Q^2)}{\pi R_A^2} = Q_s^2(x)
\]

Unitarization now results from dynamics
DIS in dipole frame

Use:
- S-matrix real
- optical theorem
- $\psi_{L,T}^\gamma \sim K_{0,1} \left(\sqrt{z(1-z)} Q |r_T| \right)$
  $\Rightarrow$ momentum scale $Q^2 \sim 1/r_T^2$
- Diffractive: $t$ is FT of $b_T$. 
Observables from dipole cross section

Predictive power!

\[\sigma_{\text{dip}}(x, r_T, \Delta_T) = \int d^2 b_T \, \frac{d^2 \sigma_{\text{dip}}(x, r_T, b_T)}{d^2 b_T} \, e^{ib_T \cdot \Delta_T}, \quad \Delta_T^2 = -t\]

\[\nabla\]

Inclusive

\[\sigma_{\gamma^*p}^{\gamma^*p} = \int d^2 r_T \int dz \, |\psi_{L,T}(Q^2, r_T, z)|^2 \sigma_{\text{dip}}(x, r_T)\]
Observables from dipole cross section

Predictive power!

\[
\sigma_{\text{dip}}(x, r_T, \Delta T) = \int d^2 b_T \frac{d^2 \sigma_{\text{dip}}(x, r_T, b_T)}{d^2 b_T} \ e^{i b_T \cdot \Delta T}, \quad \Delta T^2 = -t
\]

Inclusive \textbf{diffractive} (elastic dipole–target)

\[
\sigma^{\gamma^* p}_{L,T} = \int d^2 r_T \int dz \left| \psi_{\gamma, T}(Q^2, r_T, z) \right|^2 \sigma_{\text{dip}}(x, r_T)
\]

\[
\frac{\sigma^{D,\text{tot}}_{L,T}}{dt} = \frac{1}{16\pi} \int d^2 r_T \int dz \left| \psi_{\gamma, T}(Q^2, r_T, z) \right|^2 \sigma_{\text{dip}}^2(x, r_T, \Delta T)
\]
Observables from dipole cross section

Predictive power!

\[
\sigma_{\text{dip}}(x, r_T, \Delta_T) = \int d^2 b_T \left( \frac{d^2 \sigma_{\text{dip}}(x, r_T, b_T)}{d^2 b_T} \right) e^{i b_T \cdot \Delta_T}, \quad \Delta_T^2 = -t
\]

Inclusive diffractive (elastic dipole–target) exclusive diff.

\[
\sigma_{L,T}^{\gamma^*p} = \int d^2 r_T \int dz \left| \Psi_{L,T}^{\gamma}(Q^2, r_T, z) \right|^2 \sigma_{\text{dip}}(x, r_T)
\]

\[
\frac{\sigma_{L,T}^{D,\text{tot}}}{dt} = \frac{1}{16\pi} \int d^2 r_T \int dz \left| \Psi_{L,T}^{\gamma}(Q^2, r_T, z) \right|^2 \sigma_{\text{dip}}^2(x, r_T, \Delta_T)
\]

\[
\frac{\sigma_{L,T}^{D,V}}{dt} = \frac{1}{16\pi} \left| \int d^2 r_T \int dz \left( \Psi_{L,T}^{\gamma} \Psi_{L,T}^{\gamma*V} \right) \right|^2 \sigma_{\text{dip}}(x, r_T, \Delta_T)
\]
pA particle production, correlations

Particle production in pA is calculated in the same framework:

- DGLAP-evolved large-\(x\) pdf for proton
- CGC description of small \(x\) nucleus (unintegrated pdf)
- \(p_{\perp}\) sensitive to \(Q_s\)

\[
\frac{d\sigma_{pA \rightarrow hX}}{d^2p_T d^2b_T dy} = \int_{x_F}^1 dx f_{q/p}(x, Q^2) N_F \left( \frac{x}{x_F} \frac{p_T}{b_T} \right) D_{h/q} + \{q \leftrightarrow g\}
\]

Plot from Dumitru et al. -05
Initial conditions for AA

Classical Yang-Mills

\[ A_\mu = \text{pure gauge 1} \]

\[ A_\mu = \text{pure gauge 2} \]

\[ A_\mu = 0 \]

\[ A_{i(1,2)} = \frac{i}{g} U_{(1,2)}(x_T) \partial_i U_\dagger_{(1,2)}(x_T) \]
Initial conditions for AA

Classical Yang-Mills

Initial conditions for AA

2 pure gauges

\[ A_{(1,2)}^i = \frac{i}{g} U_{(1,2)}(x_T) \partial_i U_{(1,2)}^\dagger(x_T) \]

At \( \tau = 0 \):

\[ A_i^i \bigg|_{\tau=0} = A_{(1)}^i + A_{(2)}^i \]

\[ A^\eta \bigg|_{\tau=0} = \frac{ig}{2} [A_{(1)}^i, A_{(2)}^i] \]
Initial conditions for AA

Classical Yang-Mills

\[ \eta = \text{cst.} \]
\[ t = x^- + x^+ \]
\[ \tau = \text{cst.} \]
\[ A_\mu = \text{pure gauge 1} \]
\[ A_\mu = \text{pure gauge 2} \]
\[ A_\mu = 0 \]

At \( \tau = 0 \):

\[
A^i_\mu \bigg|_{\tau=0} = A^i_{(1)} + A^i_{(2)}
\]
\[
A^\eta \bigg|_{\tau=0} = \frac{ig}{2} [A^i_{(1)}, A^i_{(2)}]
\]

Solve numerically Yang-Mills equations for \( \tau > 0 \)
This is the glasma field.
Initial conditions for AA

**Classical Yang-Mills**

1. $A_\mu = \text{pure gauge 1}$
2. $A_\mu = \text{pure gauge 2}$
3. $\eta = \text{cst.}$
4. $A_\mu = 0$

$A_\mu = \text{?}$

$\tau = \text{cst.}$, $z$

$\sigma_{\text{dip}}$ is correlator of these Wilson lines

At $\tau = 0$:

$$A_i^i\big|_{\tau=0} = A_{(1)}^i + A_{(2)}^i$$

$$A_\eta^i\big|_{\tau=0} = ig \left[ A_{(1)}^i, A_{(2)}^i \right]$$

Solve numerically Yang-Mills equations for $\tau > 0$

This is the **glasma** field.
Kinematics

Degree of freedom $\rightarrow$ evolution
1. Color charges $\rightarrow$ JIMWLK
2. Color dipole amplitudes $\rightarrow$ BK
3. Gluons $\rightarrow$ BFKL

Historical

Logical

Kinematics:

Multi-Regge: $k_1^+ \sim p_1^+ \gg k_2^+ \sim p_2^+$

At each step $\alpha_s \ln \frac{k_i^+}{k_{i+1}^+}$

Resummed to $e^{\alpha_s \ln 1/x}$
Small $x$: the hadron/nucleus wavefunction is characterized by saturation scale $Q_s \gg \Lambda_{\text{QCD}}$. 
Small \( x \): the hadron/nucleus wavefunction is characterized by **saturation scale** \( Q_s \gg \Lambda_{\text{QCD}} \).

\[ p_T \sim Q_s : \text{strong fields } A_\mu \sim 1/g \]

- occupation numbers \( \sim 1/\alpha_s \)
- classical field approximation.
- small \( \alpha_s \), but nonperturbative
CGC

Small $x$: the hadron/nucleus wavefunction is characterized by saturation scale $Q_s \gg \Lambda_{QCD}$.

$p_T \sim Q_s$: strong fields $A_\mu \sim 1/g$

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**CGC: Effective theory for wavefunction of nucleus**

- Large $x = \text{source } \rho$, **probability** distribution $W_\gamma[\rho]$
- Small $x = \text{classical gluon field } A_\mu + \text{quantum fluctts.}$
Small $x$: the hadron/nucleus wavefunction is characterized by saturation scale $Q_s \gg \Lambda_{QCD}$.

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**CGC: Effective theory for wavefunction of nucleus**

- Large $x = \text{source } \rho$, **probability** distribution $W_y[\rho]$
- Small $x = \text{classical gluon field } A_\mu + \text{quantum fluctts.}

**Glasma** field configuration of two colliding sheets of CGC.
JIMWLK, Langevin/Diffusion picture

\[ \partial_y W_y[\rho] = \mathcal{H} W_y[\rho] \quad \rho \iff U \]

\[ \mathcal{H} \equiv \frac{\alpha_s}{2} \int_{x_T y_T z_T} \frac{\delta}{\delta \tilde{A}_c^+(y_T)} e_{T}^{ba}(x_T, z_T) \cdot e_{T}^{ca}(y_T, z_T) \frac{\delta}{\delta \tilde{A}_b^+(x_T)} , \]

\[ e_{T}^{ba}(x_T, z_T) = \frac{1}{\sqrt{4\pi^3}} \frac{x_T - z_T}{(x_T - z_T)^2} \left(1 - U^\dagger(x_T) U(z_T)\right)^{ba} \]
JIMWLK, Langevin/Diffusion picture

\[ \partial_y \mathcal{W}_y[\rho] = \mathcal{H} \mathcal{W}_y[\rho] \quad \rho \leftrightarrow U \]

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\[ \mathbf{e}_{T}^{ba}(x_T, z_T) = \frac{1}{\sqrt{4\pi^3}} \frac{x_T - z_T}{(x_T - z_T)^2} \left( 1 - U'^{(x_T)} U(z_T) \right)^{ba} \]

\[ U_{y+dy}(x_T) = U_y(x_T) e^{-i dy \alpha(x_T, y)} \]

\[ \alpha^a(x_T, y) = \sigma^a(x_T, y) + \int_{z_T} \mathbf{e}_T^{ab}(x_T, z_T) \eta_T^b(z_T, y) \]

Basis for numerical solution of JIMWLK Rummmukainen, Weigert -03
JIMWLK, Langevin/Diffusion picture

\[ \partial_y \mathcal{W}_y[\rho] = \mathcal{H} \mathcal{W}_y[\rho] \quad \rho \leftrightarrow U \]

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\[ \mathbf{e}_{T}^{ba}(x_T, z_T) = \frac{1}{\sqrt{4\pi^3}} \frac{x_T - z_T}{(x_T - z_T)^2} \left( 1 - U^\dagger(x_T) U(z_T) \right)^{ba} \]

\[ U_{y+} dy(x_T) = U_y(x_T) e^{-i dy \alpha(x_T, y)} \]

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Basis for numerical solution of JIMWLK Rummmukainen, Weigert -03

There are very few actual solutions of JIMWLK
But these are becoming important: correlations in AA
Evolution of $\mathcal{N}(x_T - y_T) = \frac{1}{N_c} \text{Tr} \left( 1 - U^\dagger(x_T)U(y_T) \right)$ from JIMWLK

$$\partial_y \left\langle \mathcal{N}(u_T) \right\rangle = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2v_T u_T^2}{v_T^2(v_T - u_T)^2} \left( \mathcal{N}(v_T) + \mathcal{N}(u_T - v_T) \right)$$

$$- \mathcal{N}(u_T) - \mathcal{N}(v_T)\mathcal{N}(u_T - v_T)$$

First equation in Balitsky hierarchy

Truncate in mean field approximation:

— justified at large $N_c$:

$$\left\langle \mathcal{N}(v_T)\mathcal{N}(u_T - v_T) \right\rangle \approx \left\langle \mathcal{N}(v_T) \right\rangle \left\langle \mathcal{N}(u_T - v_T) \right\rangle$$

When does mean field break down?

- $\left\langle \text{Tr} \ U^\dagger U \text{Tr} \ U^\dagger U \right\rangle \rightarrow$ No Weigert et al.
- $\left\langle \text{Tr} \ U^\dagger UU^\dagger U \right\rangle \rightarrow$ Maybe? Dumitru, Jalilian-Marian
Running coupling BK phenomenology

\[ \lambda = \frac{d \ln Q_s^2}{d \ln 1/x} \]

Fits to HERA data vary from \( \lambda = 0.29 \) GBW to \( \lambda = 0.18 \) bCGC.
Running coupling BK phenomenology

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The LHC points to the lower end:

McLerran, Praszalowicz
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Running \( \alpha_s \) in between Albacete

McLerran, Praszalowicz
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The LHC points to the lower end:

[Graph showing data points and fitting curves]

McLerran, Praszalowicz

Running \( \alpha_{S} \) in between Albacete

Is this the whole story?

- Running coupling \( \checkmark \) Albacete et. al.
- Full NLO Chirilli week 4
BFKL

Take also dilute limit in BK $\Rightarrow$ BFKL equation

$$
\partial_y N(u_T) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 v_T u_T^2}{v_T^2 (v_T - u_T)^2} \left( N(v_T) + N(u_T - v_T) - N(u_T) \right)
$$

Now linear like DGLAP $\Rightarrow$ can combine as DLL.

Some recent developments:

▶ Mueller-Navelet phenomenology $\Rightarrow$ Are MN jets such a good probe after all? Colferai, Schwennsen, Szymanowski, Wallon

▶ Small $x$ resummation in DGLAP $\Rightarrow$ Many advances, what is the ultimate limit of linearized approach? ABF, CCSS
**bₜ-dependence in parametrizations**

*Discussion Wed*

---

**Factorized b-dependence:**

\[
\sigma_{\text{dip}}(x, r_T, b_T) = \exp \left\{ -\frac{b_T^2}{2B} \right\} \sigma_{\text{dip}}(x, r_T) \quad \Rightarrow \quad \sigma_0 = 2\pi B
\]

Early treatments: separate \( B \) and \( \sigma_0 \) \( \Rightarrow \) inconsistent

(1 too many parameters)

---

**b-dependent saturation scale — correct unitarity limit**

- bCGC
- IPsat

Proton grows like \( b^2 \sim \ln r \)

(Is this problem?)
\( b_T \)-dependence in actual BK?

Coulomb tails in vacuum lead to unphysical growth of the proton with energy.

E.g. Golec-Biernat, Stasto -03

Interface of confinement and saturation, hard!
$b_T$-dependence in actual BK?

Coulomb tails in vacuum lead to unphysical growth of the proton with energy.

E.g. Golec-Biernat, Stasto -03

Interface of confinement and saturation, hard!

What to do?

- $b$-indep. BK for different $b$ separately? Levin, Lublinsky et al.
- $Q_s(x) \rightarrow Q_s(x, t)$; $b$-space? Marquet, Peschanski, Soyez
- Impose $\Lambda_{QCD}$ cutoff in evolution Avsar et al.
- ? New ideas
$b_T$—dependence of different processes, $Q_s(b)$

Dominant impact parameters different

\[ b_{\text{diff}}(q\bar{q}) < b_{\text{incl}} < b_{\text{diff}}(q\bar{q}g) \]

Integrand vs. $b_T$ for

- $q\bar{q}$
- Inclusive
- $q\bar{q}g$

\[ Q^2 = 100 \text{GeV}^2 \]

\[ (x_P = 10^{-3}) \]
$b_T$-dependence in nuclei, beyond Glauber?

Evolution and independent $\gamma^* N$ scatterings don’t commute.

\[ p : \quad S_p(b_T) \sim e^{-Q_s^p(b_T)^2 r^2} \]

\[ A : \quad S_A(b_T) \sim e^{-AT_A(b_T) Q_s^p r^2} \]

Origin of $\perp$ shape confinement,
HE evolution weak coupling:
how to combine?
$b_T$-dependence in nuclei, beyond Glauber?

Evolution and independent $\gamma^* N$ scatterings don’t commute.

Evolution $x \to 0$

\[
p: \quad S_p(b_T) \sim e^{-Q_p^p(b_T)^2 r^2} \quad \Rightarrow \quad S_p(b_T) \sim e^{-(Q_p^p(b_T)^2 r^2)\gamma}
\]

\[
A: \quad S_A(b_T) \sim e^{-AT_A(b_T)Q_s^p r^2} \quad \Rightarrow \quad S_A(b_T) \sim e^{-(AT_A(b_T)Q_s^p r^2)\gamma}
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Origin of $\perp$ shape confinement,

HE evolution weak coupling:

how to combine?
Conclusions

I talked about:

- Small $x$ and coherence
- Predictive power of dipole picture
- Evolution equations
- Impact parameter dependence

I am open to questions, but the audience here is probably more qualified to answer them than I am...