QCD RESUMMATION
(LARGE X)

Eric Laenen

University of Amsterdam  NIKHEF  Universiteit Utrecht

INT workshop on Gluon and Quark Sea at High Energies, Seattle, Sep. 13-29, 2010
OUTLINE

- Basic issues, selected results
- Some new (eikonal) developments
  - Next-to-eikonal corrections
  - Path-integral methods, webs for multiple parton scattering
- Summary
RESUMMATION

- Catch-all phrase for summing something to all orders
  - why would one do that?
  - what can one sum?
  - when should one do that?
  - to what accuracy?
- Resummation:
  
  *The art of constructing for some quantity out of a subset of perturbative terms, aided by extra insight, an all-order expression*
WHY WOULD ONE RESUM?

- Rescue predictive power
  - when perturbative series converges poorly
  - Predicts terms in next order (approximate NNLO)
- Better physics description
- Less scale uncertainty

CAVEATS

- New uncertainty from IR renormalons
- Sloppy definitions, casual use
PREDICTIVE POWER

\[ \hat{O} = \sum_{n} c_n \alpha^n + R_n \]

- \( c_n \) computed with Feynman diagrams
- Finite order: only take lowest few \( "n" \). Check if
  - \( \alpha \) is small enough?
    - (power correction is \( R_n \) small enough ?)
  - \( c_n \) does not grow too fast with \( n \)?
QCD HADRONIC OBSERVABLES

- Use equation in multiple ways
  - Find weakest link, and update
    - Predict $O$, or fit $\varphi$ or infer $P$
  - But $c_n$ often misbehave
    - grow systematically with $n$

$$ O = \phi \otimes \hat{O} + P_0 $$

$PDF's$

$Power\ correction$

$Partonic\ observable$

$$ \hat{O} = \sum_n c_n \alpha^n + R_n $$
Typical behavior of cross section

\[ \hat{O}_2 = 1 + \alpha(L^2 + L + 1) + \alpha^2(L^4 + L^3 + L^2 + L + 1) + \ldots \]

“1” = \( \pi^2, 2, \) etc

Effective expansion parameter is \( \alpha L^2 \)

Fix: reorganize / resum such that

\[ \hat{O} = 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \ldots \]

\[ = \exp \left( \sum_{LL} L g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots \right) C(\alpha_s) \text{ constants} \]

+ suppressed terms

Just as systematically improvable as PT

- LO, NLO \( \rightarrow \) LL, NLL, etc
RESUMMATION OF WHAT?

- So many logs...

\[ \ln(1 - T), \ln\left(\frac{p_T}{m_Z}\right), \ln\left(\frac{1}{x}\right), \ln\left(\frac{kT}{x}\right), \ln\left(\frac{\mu}{\nu}\right), \ln(1 - T), \ln(N), \ln(b) \]

- Many, many results
RECOIL DOUBLE LOGS

- Eg. $p_T$ of Z-bosons produced at Tevatron

\[ L^2 = \ln^2 \left( \frac{p_T^2}{M_Z^2} \right) \]

- Visible logs

- Z-boson get $p_T$ from recoil against (soft) gluons
Invisible logs can also plague perturbation theory

\[ S \gtrsim Q^2 \quad \rightarrow \quad \frac{Q^2}{s} \approx 1 \quad \rightarrow \quad \ln^2 \left( 1 - \frac{Q^2}{s} \right) \gg 1 \]
RESUMMATION IN A FEW EASY STEPS

- Cross section for n extra gluons

\[ \sigma(n) = \frac{1}{2s} \int d\Phi_{n+1}(P, k_1, \ldots, k_n) \times |M(P, k_1, \ldots, k_n)|^2 \]

- When emissions are soft, approximate phase space measure and matrix element

\[ d\Phi_{n+1}(P, k_1, \ldots, k_n) \rightarrow d\Phi(P) \times \left( d\Phi_1(k) \right)^n \frac{1}{n!} \quad |M(P, k_1, \ldots, k_n)|^2 \rightarrow |M(P)|^2 \times |M_{1\text{ emission}}(k)|^2 \]

- Sum over all orders

\[ \sum_n \sigma(n) = \sigma(0) \times \exp \left[ \int d\Phi_1(k) |M_{1\text{ emission}}(k)|^2 \right] \]

- Incorporate Theta or Delta functions for specific cases

- but these must factorize similarly, or they cannot go into exponent
PHASE SPACE CONSTRAINTS

- Kinematic condition expresses "z" in terms of gluon energies

\[ s = Q^2 - 2P \cdot K - K^2 \quad \delta \left( 1 - \frac{Q^2}{s} - \sum_i \frac{2k_i^0}{\sqrt{s}} \right) \]

- Transform factorizes the phase space

\[ \int_0^\infty dw \, e^{-wN} \delta \left( w - \sum_i w_i \right) = \prod_i \exp(-w_iN) \]

- So can go into exponent

\[ \sum_n \sigma(n) = \sigma(0) \times \exp \left[ \int d\Phi_1(k)|M_{1\text{ emission}}(k)|^2(\exp(-wN) - 1) \right] \]

- Large logs: \( \log(N) \)
ON THE ORIGIN OF logs

- Logarithms in cross sections are related to IR divergences

\[
\frac{1}{(p + k)^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_g E_q(1 - \cos\theta_{qg})}
\]

Phase space integration

\[
\alpha_s \int \frac{d^4-2\epsilon k}{(2\pi)^4} \frac{p \cdot p'}{p \cdot k p' \cdot k} \sim \alpha_s \int K \frac{dE_g E_g^{-\epsilon}}{E_g} \int d\theta_{qg} \sin^{-\epsilon} \theta_{qg} \sim \alpha_s \left( \frac{1}{\epsilon^2} + \ln^2(K) \right).
\]

- When one can find all IR divergences, one can find all the logs (and a good number of constants)
Fix both \( k_T \) and energy in exponent

- Can make \( k_T \) integral visible (Higgs, Vector Boson)
- Can make \( k_T \) integral invisible (Prompt photon, heavy quark, sleptons)

\[
\frac{d\sigma}{dp_T} = C \sum_{a,b} \int \frac{dN}{2\pi i} \phi_a(N)\phi_b(N) \int dx_T^2 (x_T^2)^N \int d^2b \int d{k_T,a} \int d{k_T,b} \times
\delta(Q_T - k_{T,a} - k_{T,b})e^{-ib\cdot Q_T} \left( \frac{S}{4(p_T - Q_T/2)^2} \right)^N \exp \left[ E(N, b, p_T) \right]
\]

- Key ingredient: PDF at fixed energy and \( k_T \)

\[
R(z, k_T) = c \int \frac{d\lambda}{2\pi} \frac{d^2b}{(2\pi)^2} e^{-i\lambda z p^0 + ib\cdot k_T} \langle p|\bar{\psi}(\lambda, b)\gamma^+ \psi(0)|p\rangle
\]
- Matching to TMD’s? Or to NLO?
RESUMMATION AND FACTORIZATION

- Very generically, if a quantity factorizes, one can resum it
  - Renormalization; factorizes UV modes \( G_B(g_B, \Lambda, p) = Z \left( \frac{\Lambda}{\mu}, g_R(\mu) \right) \times G_R \left( g_R(\mu), \frac{p}{\mu} \right) \)
  - Evolution equation \( \mu \frac{d}{d\mu} \ln G_R \left( g_R(\mu), \frac{p}{\mu} \right) = -\mu \frac{d}{d\mu} \ln Z \left( \frac{\Lambda}{\mu}, g_R(\mu) \right) = \gamma(g_R(\mu)) \)
  - Solving = resummation \( G_R \left( \frac{p}{\mu}, g_R(\mu) \right) = G_R \left( 1, g_R(p) \right) \exp \left[ \int_{p}^{\mu} \frac{d\lambda}{\lambda} \gamma(g_R(\lambda)) \right] \)

- Type of factorization dictates resummation
  - small \( x \rightarrow k_T \) factorization
  - large \( x \rightarrow \) near-threshold factorization

- Factorization is essentially separating degrees of freedom
  - Systematic approach in Soft Collinear Effective Theory

Talk by Iain Stewart
Threshold-resummed Drell-Yan cross section

\[ \frac{d\sigma_{\text{resum}}}{dQ^2}(z) = \int_C \frac{dN}{2\pi i} z^{-N} \hat{\sigma}(N) \]

\[ \sigma(N) = \exp \left[ - \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \left\{ \int_{Q^2}^{Q^2(1-x)^2} \frac{d\mu}{\mu} A(\alpha_s(\mu)) + D(\alpha_s((1-x)Q)) \right\} \right] \times (1 + \alpha_s(Q^2) \frac{C_F}{\pi} + \ldots) \]

\[ \hat{\sigma}_{DY}(N, Q^2) = g_0(Q^2) \exp [G_{DY}^N(Q^2)] \]

\[ G_{DY}^N = \ln Ng_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \ldots, \quad \lambda = \beta_0 \alpha_s \ln N \]

Good convergence in exponent
NNNLL THRESHOLD DY AND HIGGS

\[ D(\alpha_s) = 4 \quad B_\delta(\alpha_s) = -2 \quad G(\alpha_s) = \frac{d}{d\alpha_s} F_{\text{MS}}(\alpha_s) \]

Single logs \quad virtual split fn. \quad form factor

Moch, Vermaseren, Vogt
SOME RULES OF THUMB

Why increases in QCD resummation?

\[ \sigma_{\text{partonic, resum}}(N) = \frac{\sigma_{\text{hadronic}}(N)}{\phi^2(N)} = \frac{\exp(-\ln^2 N)}{(\exp(-\ln^2 N))^2} = \exp(+\ln^2 N) \]

Why scale cancellation?

\[ \phi(N) \sim \exp[(A \ln N + B)\ln(\mu_F)] \]

\[ \sigma_{\text{partonic, resum}}(N) \sim \exp[E(\ln^2 N) - \ln \mu_F (A \ln N + B)] \]

Sterman, Vogelsang
For DY, DIS, Higgs, singular behavior when $x \to 1$

$$\delta(1-x) \left[ \frac{\ln^i(1-x)}{1-x} \right] \ln^k(1-x)$$

- singularity structure for plus distributions is organizable to all orders, perhaps also for divergent logarithms?

- After Mellin transform

$$\text{Constants} \quad \ln^i(N) \quad \frac{\ln^k(N)}{N}$$

- We know a lot about logs and constants, very little about $1/N$
**LN(N) / N TERMS**

- Can be numerically important
  - Kraemer, EL, Spira

- We know that the leading series ln^i(N) / N exponentiates
  - by replacing in resummation formula

\[
\frac{1 + z^2}{1 - z} \rightarrow \frac{2}{1 - z} - 2
\]
Moch, Vermaseren, Vogt noted a remarkable relation

\[ \gamma_{qq}(N) = A(\alpha_s) \ln N + B(\alpha_s) + C(\alpha_s) \frac{\ln N}{N} + \ldots \]

- DMS reproduced this by changing DGLAP equation

\[
\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi \left( \frac{x}{z}, z\mu^2 \right) \mathcal{P} \left( z, \alpha_s \left( \frac{\mu^2}{z} \right) \right)
\]

\[
\mathcal{P} \left( z, \alpha_s \right) = \frac{A(\alpha_s)}{(1 - z)_+} + B_5(\alpha_s) \delta(1 - z) + O ((1 - z))
\]

- Can this be reproduced in threshold resummation?
EXTENDED THRESHOLD RESUMMATION

Ansatz: modified resummed expression

\[ \ln \left[ \sigma(N) \right] = \mathcal{F}_{\text{DY}}(\alpha_s(Q^2)) + \int_0^1 dz z^{N-1} \left\{ \frac{1}{1-z} D \left[ \alpha_s \left( \frac{(1-z)^2 Q^2}{z} \right) \right] + 2 \int_{Q^2} (1-z)^2 Q^2/z dz \right\} \]

where

\[ P_s^{(n)}(z) = \frac{z}{1-z} A^{(n)} + C_\gamma^{(n)} \ln(1-z) + D_\gamma^{(n)} \]

(We constructed a similar expression for DIS). Structure:

\[ \sigma(N) = \sum_{n=0}^{\infty} \left( g^2 \right)^n \left[ \sum_{m=0}^{2n} a_{nm} \ln^m N + \sum_{m=0}^{2n-1} b_{nm} \frac{\ln^m N}{N} \right] + \mathcal{O} \left( N^{-2} \right) \]

<table>
<thead>
<tr>
<th>( C_F^2 )</th>
<th>( C_A C_F )</th>
<th>( n_f C_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{23} )</td>
<td>(4)</td>
<td>(7)</td>
</tr>
<tr>
<td>( b_{22} )</td>
<td>(8\zeta_2 - \frac{43}{4})</td>
<td>(8\zeta_2 - 11)</td>
</tr>
<tr>
<td>( b_{21} )</td>
<td>(-\frac{1}{2}\zeta_2 - \frac{3}{4})</td>
<td>(4\zeta_2)</td>
</tr>
<tr>
<td>( b_{20} )</td>
<td>(-\frac{1}{2}\zeta_2 - \frac{3}{4})</td>
<td>(8\zeta_2 - 11)</td>
</tr>
</tbody>
</table>

Must understand systematics beyond eikonal approximation
Soft emission by charged particle

- **Propagator**: expand numerator & denominator in soft momentum, keep lowest order
- **Vertex**: expand in soft momentum, keep lowest order

\[
\frac{(p + k)^\mu + p^{\mu}}{2p \cdot k + k^2} \rightarrow \frac{2p^\mu}{2p \cdot k}
\]
Exact: \[
\frac{1}{(p + K_1)^2} (2p + K_2 + K_1)^{\mu_1} \cdots \frac{1}{(p + K_n)^2} (2p + K_n)^{\mu_n}, \quad K_i = \sum_{m=i}^{n} k_m.
\]

Approx: \[
\frac{1}{2pK_1} 2p^{\mu_1} \cdots \frac{1}{2pK_n} 2p^{\mu_n}
\]

Eikonal identity: \[
\frac{1}{p \cdot (k_1 + k_2)} p \cdot k_2 + \frac{1}{p \cdot (k_1 + k_2)} p \cdot k_1 = \frac{1}{p \cdot k_1 p \cdot k_2}
\]

Sum over all perm’s: \[
\prod_{i} \frac{p^{\mu_i}}{p \cdot k_i}.
\]

Independent, uncorrelated emissions, Poisson process.
Non-Abelian Eikonal Approximation

- Same methods as for QED, but organization harder: SU(3) generator at every vertex
  - no obvious decorrelation

- Key “object”: Wilson line
  \[ \Phi_n(\lambda_2, \lambda_1) = P \exp \left[ ig \int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A^a(\lambda n) T_a \right] \]
  - Order by order in “g”, it generates QCD eikonal Feynman rules

Order the \( T_a \) according to \( \lambda \)
One loop vertex correction, in eikonal approximation

$$A_0 \int d^nk \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation

$$A_0 \frac{1}{2} \left( \int d^nk \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

Exponential series

$$\exp \left[ \right]$$
Take quark - antiquark line, connect with soft gluons in all possible ways, use eikonal approximation.

Exponentiation still occurs, without path ordering!

- A selection of diagrams in exponent, but with modified color weights: “webs”

- Webs are two-eikonal line irreducible

- Proof by induction; recursive definition of color weights

- How can we extend this to include next-to-eikonal terms?
Represent propagator as particle path integral, between coord. and momentum states

\[
\tilde{\Delta}_F(p_f^2) = \frac{1}{2} \int_0^\infty dT \frac{\langle p_f | U(T) | x_i \rangle}{\langle p_f | x_i \rangle} = -\frac{i}{p_f^2 + m^2 - i\varepsilon}
\]

where

\[
\langle p_f | U(T) | x_i \rangle = e^{-ip_f x_i - i\frac{1}{2}(p_f^2+m^2)T} \int_{x(0)=0}^{p(T)=0} \mathcal{D}p \mathcal{D}x \ e^{i \int_0^T dt (p \dot{x} - \frac{1}{2}p^2)}
\]

Add an (abelian) gauge field

\[
\langle p_f | U(T) | x_i \rangle = \int_{x(0)=x_i}^{p(T)=p_f} \mathcal{D}p \mathcal{D}x \exp \left[ -ip(T)x(T) + i \int_0^T dt (p \dot{x} - \frac{1}{2}(p^2 + m^2) + p \cdot A + i \frac{1}{2} \partial \cdot A - \frac{1}{2} A^2) \right]
\]

n-point Green’s function

\[
G(p_1, \ldots, p_n) = \int \mathcal{D}A_s^\mu H(x_1, \ldots, x_n) \times \langle p_1 | ((p - A_s)^2 - i\varepsilon)^{-1} | x_1 \rangle \ldots \langle p_n | (p - A_s)^2 - i\varepsilon)^{-1} | x_n \rangle
\]
Truncate external lines for S-matrix element

\[ i(p_f^2 + m^2)|p_f| - i((p - A)^2 - i\varepsilon)^{-1}|x_i| = e^{-i p_f x_i} f(\infty) \]

\[ S(p_1, \ldots, p_n) = \int \mathcal{D}A_s H(x_1, \ldots, x_n) e^{-i p_1 x_1} f_1(\infty) \ldots e^{-i p_n x_n} f_n(\infty) e^{i S[A_s]} \]

\[ f(\infty) = \int_{x(0)=0} dx e^{i \int_0^\infty dt \left( \frac{1}{2} \dot{x}^2 + (p_f + \dot{x}) \cdot A(x_i + p_f t + x(t)) + i \partial \cdot A(x_i + p_f t + x) \right)} \]

Eikonal vertices act as sources for gauge bosons along path

Disconnected

Connected

QED: exponentiation now textbook result:
all diagrams = exp (connected diagrams)
REPLICA TRICK

- Can relate exponentiation of soft gauge fields to that of connected diagrams in QFT. Proof: replica trick (from stat. mech.)

- Consider a $N$ copies of a scalar theory

$$Z[J]^N = \int D\phi_1 \ldots D\phi_N e^{i S[\phi_1] + \ldots + i S[\phi_N] + J\phi_1 + \ldots J\phi_N}$$

- If $Z$ is exponential, find out what contributes to $\log Z$

$$Z^N = 1 + N \log Z + \mathcal{O}(N^2)$$

- Amounts to diagrams that allow only one replica $\rightarrow$ connected!

\[\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}\]
REPLICA TRICK

- Can relate exponentiation of soft gauge fields to that of connected diagrams in QFT. Proof: replica trick (from stat. mech.)

- Consider a N copies of a scalar theory

\[ Z[J]^N = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_N e^{iS[\phi_1] + \ldots + iS[\phi_N] + J\phi_1 + \ldots J\phi_N} \]

- If Z is exponential, find out what contributes to log Z

\[ Z^N = 1 + N \log Z + \mathcal{O}(N^2) \]

- Amounts to diagrams that allow only one replica → connected!
Amplitude for two colored lines

\[ S(p_1, p_2) = H(p_1, p_2) \int DA_s f(\infty) e^{iS[A_s]} \]

Replicate, and introduce ordering operator

\[ f(\infty) = \mathcal{P} \exp \left[ \int dx \cdot A(x) \right] \quad \prod_{i=1}^{N} \mathcal{P} \exp \left[ \int dx \cdot A_i(x) \right] = \mathcal{R}\mathcal{P} \exp \left[ \sum_{i=1}^{N} \int dx \cdot A_i(x) \right] \]

Look for diagrams of replica order \( N \). These will go into exponent

(a) is order \( N \)

(b) for equal replica number \( (i=j) \): \( C_F^2 \). For \( i \neq j \) also \( C_F^2 \). Sum: \[ NC_F^2 + N(N - 1)C_F^2 = N^2 C_F^2 \]

(c) for equal replica number \( (i=j) \): \( C_F^2 - C_F C_A / 2 \).

For \( i \neq j \) \( C_F^2 \). Term linear in \( N \):

\[ N \left( C_F^2 - \frac{C_F C_A}{2} \right) + (-N)C_F^2 = N \left( -\frac{C_F C_A}{2} \right) \]
Amplitude for two colored lines

\[ S(p_1, p_2) = H(p_1, p_2) \int D A_s f(\infty) e^{iS[A_s]} \]

Replicate, and introduce ordering operator

\[ f(\infty) = P \exp \left[ \int dx \cdot A(x) \right] \prod_{i=1}^{N} P \exp \left[ \int dx \cdot A_i(x) \right] = R P \exp \left[ \sum_{i=1}^{N} \int dx \cdot A_i(x) \right] \]

Look for diagrams of replica order N. These will go into exponent

(a) is order N
(b) for equal replica number (i=j): \( C_F^2 \). For i≠j also \( C_F^2 \). Sum: \( N C_F^2 + N(N-1)C_F^2 = N^2 C_F^2 \)
(c) for equal replica number (i=j): \( C_F^2 - C_F C_A /2 \). For i≠j \( C_F^2 \). Term linear in N:

\[ N \left(C_F^2 - \frac{C_F C_A}{2}\right) + (-N)C_F^2 = N \left(-\frac{C_F C_A}{2}\right) \]
Can give an all-order, non-recursive formula for modified color factors

- Consider general diagram $G$, with a number of connected pieces

- Distribute replica numbers in all possible ways, and count for each such partition $P$ the multiplicity to linear order in $N$

$$\overline{C}(G) = \sum_P (-)^{n(P)-1} (n(P) - 1)! \prod_g C(g)$$

- Examples

$$\overline{C}(X) = C(X) - C(I)C(I)$$

$$= (C_F^2 - \frac{1}{2} CA CF) - C_F^2$$

$$\overline{C}(IX) = C(IX) - C(I)C(X) = 0$$
Wilson lines are classical solutions of path integral

Fluctuations around classical path lead to NE corrections

- This class of NE corrections exponentiates
- Keep track via scaling variable \( \lambda \) \( p^\mu = \lambda n^\mu \)

\[
f(\infty) = \int_{x(0)=0} Dx \exp \left[ i \int_0^\infty dt \left( \frac{\lambda}{2} \dot{x}^2 + (n + \dot{x}) \cdot A(x_i + nt + x) + \frac{i}{2\lambda} \partial \cdot A(x_i + p_f t + x) \right) \right]
\]

Use 1-D field theory propagators

\[
\langle x(t)x(t') \rangle = G(t, t') = \frac{i}{\lambda} \min(t, t')
\]

\[
\langle \dot{x}(t)\dot{x}(t') \rangle = \frac{i}{\lambda} \delta(t - t')
\]
\[ f(\infty) = \exp \left[ -\int \frac{d^d k}{(2\pi)^d} \frac{n^\mu}{n \cdot k} \tilde{A}_\mu(k) + \frac{1}{2\lambda} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{n \cdot k} \tilde{A}_\mu(k) + \sum \rightarrow + \sum \bigcirc \right] \]

Both 2-point correlators and tadpoles contribute

NE Feynman rules
One soft emission determined by elastic amplitude to eikonal and next-to-eikonal order

\[ \Gamma^\mu = \Gamma^{\mu} + \Gamma^{\mu} + \Gamma^{\mu} \]

\[ \Gamma^\mu = \left[ \frac{(2p_1 - k)^\mu}{-2p_1 \cdot k} + \frac{(2p_2 + k)^\mu}{2p_2 \cdot k} \right] \Gamma + \left[ \frac{p_1^\mu (k \cdot p_2 - k \cdot p_1)}{p_1 \cdot k} + \frac{p_2^\mu (k \cdot p_1 - k \cdot p_2)}{p_2 \cdot k} \right] \frac{\partial \Gamma}{\partial p_1 \cdot p_2} \]

Analyzed in context of jet-soft factorization by Del Duca

One emission from H still missing in our approach
Path integral method provides elegant way to derive Low’s theorem

\[ S(p_1, \ldots, p_n) = \int \mathcal{D}A_s H(x_1, \ldots, x_n; A_s) e^{-i p_1 x_1} f(x_1, p_1; A_s) \cdots e^{-i p_n x_n} f(x_1, p_1; A_s) e^{i S[A_s]} \]

Gauge transformation must cancel between f’s and H

\[ f(x_i, p_f; A) \rightarrow f(x_i, p_f; A + \partial \Lambda) = e^{-iq\Lambda(x_i)} f(x_i, p_f; A) \]

Opposite transformation in H, expand to first order in A and \( \Lambda \)

Low’s contribution is then:

\[ S(p_1, \ldots, p_n) = \int \mathcal{D}A \left[ \int \frac{d^d k}{(2\pi)^d} \sum_j^n q_j \left( \frac{n_j^\mu}{n_j \cdot k} k_\nu \frac{\partial}{\partial p^\nu_{j\mu}} - \frac{\partial}{\partial p^\mu_{j\nu}} \right) H(p_1, \ldots, p_n) A_\mu(k) \right] \times f(0, p_1; A) \cdots f(0, p_n; A) \]

First term is due to displacement of \( f(x,p,A) \)

Analogous result in non-abelian case, for \( n=2 \)
Exponentiation of soft emissions for matrix elements as “connected diagrams”

- For both eikonal and next-to-eikonal contributions from external lines
- Replica trick both for exponentiation, and for explicit expression for webs. New NE Webs.

- 1 emission from hard part also included
- QCD: 2 lines ok. 3 lines also easy. 4+ (later)

Can we arrive at the same results using diagrams, and inductive reasoning?

- Combinatorics challenging
Recall: Abelian case, multiple emission, and sum over permutations

Eikonal identity:
\[ \frac{1}{p \cdot (k_1 + k_2)p \cdot k_2} + \frac{1}{p \cdot (k_1 + k_2)p \cdot k_1} = \frac{1}{p \cdot k_1 p \cdot k_2} \]

For many emissions
\[ \prod_i \frac{p^{\mu_i}}{p \cdot k_i} \]

Non-abelian case requires
- web: two-eikonal irreducible graph
- “group”: projection of web on external line
- analogue of eikonal identity for permutations that leaves ordering in group invariant

\[ \sum_{\pi} \frac{1}{2p \cdot k_{\pi_1}} \frac{1}{2p \cdot (k_{\pi_1} + k_{\pi_2})} \cdots \frac{1}{p \cdot (k_{\pi_1} + \ldots + k_{\pi_n})} = \prod_g \frac{1}{2p \cdot k_{g_1}} \frac{1}{2p \cdot (k_{g_1} + k_{g_2})} \cdots \frac{1}{2p \cdot (k_{g_1} + \ldots + k_{g_m})} \]
Can also use in reverse, as “merging”

\[ E(H_1) E(H_2) = \sum_{\pi_A} \sum_{\pi_B} E(H_1 \cup_{\pi_A} \pi_B H_2) \]

Collect identical diagrams

\[ E(H_1) E(H_2) = \sum_G E(G) N_{G|H_1 H_2} \]

\[
\exp \left\{ \sum_i \tilde{c}_H E(H) \right\} = \prod_H \left( \sum_{n} \frac{1}{n!} \left[ \tilde{c}_H E(H) \right]^{n} \right) = \sum_G c_G E(G)
\]

Prove that, for normal color factors on rhs, those on left side are those of webs

Proof uses

• induction
• combinatorics
• simplicity of color structure
- Identify next-to-eikonal vertices
  - show that they “decorrelate”, once summed over all perm’s. Use induction again
    - as eikonal webs, but now with a special vertex
    - for fermions: they become spin-sensitive
  - new correlations between eikonal webs $\rightarrow$ NE webs
Check use of NE Feynman rules for Drell-Yan double real emission

- for amplitude, expand

\[ \mathcal{A} = \mathcal{A}^{(0)} \exp \left[ \mathcal{A}^{(1)}_{\text{E}} + \mathcal{A}^{(1)}_{\text{NE}} + \mathcal{A}^{(2)}_{\text{NE}} \right] \]

- to 2nd order, and integrate each term with exact 3-particle phase space. Cross term leads to

\[
\left( \frac{\alpha_s C_F}{4\pi} \right)^2 \left[ -\frac{1024 \log^3(1 - z)}{3} - \frac{512 \log^2(1 - z)}{e} - \frac{512 \log(1 - z)}{e^2} - \frac{256}{e^3} \right]
\]

- Also need special vertex

\[
R^\mu\nu(p; k_1, k_2) = \frac{- (p \cdot k_2) p^\mu k_1^\nu + (p \cdot k_1) k_2^\mu p^\nu - (p \cdot k_1)(p \cdot k_2) g^\mu\nu - (k_1 \cdot k_2) p^\mu p^\nu}{p \cdot (k_1 + k_2)}
\]
DRELL-YAN CHECK

- Combine with exact phase space

\[ K^{(2)NE} = \left( \frac{\alpha_s C_F}{4\pi} \right)^2 \left[ \frac{1024\mathcal{D}_3}{3} - \frac{1024 \log^3(1 - z)}{3} + 640 \log^2(1 - z) \right. \]
\[ + \frac{512\mathcal{D}_2 - 512 \log^2(1 - z) + 640 \log(1 - z)}{\epsilon} + \left. \frac{512\mathcal{D}_1 - 512 \log(1 - z)}{\epsilon^2} \right. \]
\[ + \frac{256\mathcal{D}_0 - 256}{\epsilon^3} \]

\[ \mathcal{D}_i = \left[ \frac{\log^i(1 - z)}{1 - z} \right]_+ \]

- Agrees with equivalent exact result
Replica trick for multiple colored lines; find again order N terms

- even when “present”, these may be kinematically zero

\[ Z_{I,J}^N = \int D\mathbf{A}^1 \ldots D\mathbf{A}^N e^{\sum S[A^i]} \times \left[ (W_1^{(1)} \ldots W_N^{(1)}) \ldots \right] \]
We find

\[ \sum \mathcal{F}(D)C(D) = \exp[\sum_{d,d'} \mathcal{F}(d)R_{dd'} C(d')] \]

Example: “closed sets” of diagrams

\[ \frac{1}{6} \left[ C(3a) - C(3b) - C(3c) + C(3d) \right] \times \left[ M(3a) - 2M(3b) - 2M(3c) + M(3d) \right] \]

Closed form solution for modified color factor

\[ \overline{C}(G) = \sum_P \left( \frac{(-)^{n(P)-1}}{n(P)} \right) \sum_{\pi} C(g_{\pi_1}) \cdots C(g_{\pi_{n(P)}}) \]
SUMMARY

- QCD (large-x) resummation consistent framework, generally successful
- Eikonal approximation important ingredient
- Next-to-eikonal contributions form new webs, and exponentiate
  - using path integrals, or diagrams
    - Feynman rules for exponent of scattering amplitude
  - classified “Low’s theorem” contributions
- Multi-parton webs defined
- To do: application to cross sections