Topological aspects of high-energy QCD

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Outline

1. High orders of perturbation theory and classical solutions

2. Gauge fields and topology

3. Topological effects in high-energy QCD

4. Chiral Magnetic Effect and charge asymmetries@RHIC

5. Topological effects in the fragmentation of polarized quarks and in SIDIS: Belle, RHIC, JLab, EIC
High orders of perturbation theory and classical solutions

Is the perturbative expansion (PE) convergent?

consider a PE of some amplitude:

\[ F(e^2) = a_0 + a_2 e^2 + a_4 e^4 + \ldots \]

If this series were convergent, F would be an analytical function of e at e=0, and so F(-e^2) would also converge.

But: e^2 < 0 describes the world where the like charges attract each other. The vacuum in this world is unstable: it can decay to a ground state where the electrons and positrons occupy different regions of space; such a decay would involve a large number of particles.

Because of this, the amplitude is likely to have a divergence at orders \( \sim 1/e^2 \)
High orders of perturbation theory and classical solutions


Lipatov demonstrated explicitly that the divergence of perturbative expansion in scalar theories with dimensionless negative coupling is determined by spherically symmetrical classical Euclidean solutions. This analysis has been extended to Yang-Mills theories coupled to scalar fields - high orders are dominated by instantons.

For the same reason, classical solutions dominate large orders of perturbation theory if the expansion is performed around an unstable vacuum.

The vacuum of perturbative QCD is unstable (due to the asymptotic freedom)

\[
\text{Re} V_{\text{pert}}(H) = \frac{1}{2} H^2 + (gH)^2 \frac{b}{32 \pi^2} \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} \right)
\]
High orders of perturbation theory and classical solutions

The behavior of perturbation theory in QCD at high orders is very likely connected to the classical solutions.

Classical solutions in QCD are linked to topology of gauge fields.
Gauge fields and topology

Möbius strip, the simplest nontrivial example of a fiber bundle

Gauge theories “live” in a fiber bundle space that possesses non-trivial topology (knots, links, twists,...)

NB: Maxwell electrodynamics as a curvature of a line bundle
Characteristic forms and geometric invariants

By Shiing-shen Chern and James Simons*

1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one could evaluate these boundary integrals, add up over the triangulation, and have the geometry wash out, leaving the sought after combinatorial formula. This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right and it and its generalization are the subject of this paper.
6. Applications to 3-manifolds

In this section $M$ will denote a compact, oriented, Riemannian 3-manifold, and $F(M) \to M$ will denote its $SO(3)$ oriented frame bundle equipped with the Riemannian connection $\theta$ and curvature tensor $\Omega$. For $A, B$ skew symmetric matrices, the specific formula for $P_i$ shows $P_i(A \otimes B) = -(1/8\pi^2) \text{tr} AB$. Calculating from (3.5) shows

$$TP_1(\theta) = \frac{1}{4\pi^2} \{ \theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23} \}.$$
Chern-Simons theory

What does it mean for a gauge theory?

**Geometry**
- Riemannian connection
- Curvature tensor

**Physics**
- Gauge field
- Field strength tensor

\[
S_{CS} = \frac{k}{8\pi} \int_M d^3x \, \epsilon^{ijk} \left( A_i F_{jk} + \frac{2}{3} A_i [A_j, A_k] \right)
\]

Abelian non-Abelian
Chern-Simons theory

\[ S_{CS} = \frac{k}{8\pi} \int_M d^3x \, \epsilon^{ijk} \left( A_i F_{jk} + \frac{2}{3} A_i [A_j, A_k] \right) \]

Remarkable novel properties:

- gauge invariant, up to a boundary term

- topological - does not depend on the metric, knows only about the topology of space-time \( M \)

- when added to Maxwell action, induces a mass for the gauge boson - different from the Higgs mechanism!

- breaks Parity invariance
Chern-Simons theory and the vacuum of Quantum Chromodynamics

Equation:

\[ D^\mu F^{a\mu\nu} = 0 \]

Solution:

\[ A^a_\mu(x) = \frac{2\eta_{a\mu\nu}x_\nu}{x^2 + \rho^2} \]

Integer

\[ Q = \int d\sigma_\mu K_\mu \]

\[ K_\mu = \frac{1}{16\pi^2} \epsilon_{\mu\alpha\beta\gamma} \left( A^a_\alpha \partial_\beta A^a_\gamma + \frac{1}{3} f^{abc} A^a_\alpha A^b_\beta A^c_\gamma \right) \]

Chern-Simons current

Coupling of space-time and color:

\[ \eta_{a\mu\nu} = \begin{cases} 
\epsilon_{a\mu\nu} & \text{if } \mu, \nu = 1, 2, 3, \\
\delta_{a\mu} & \text{if } \nu = 4, \\
-\delta_{a\nu} & \text{if } \mu = 4. 
\end{cases} \]

Belavin, Polyakov, Tyupkin, Schwartz; 't Hooft; ...
Topography-induced change of chirality

Right ↔ Left

Color
SU(2) spin

Spin

Momentum

\[ \vec{J} = \vec{T} + \vec{S} \]
Topological number fluctuations in QCD vacuum
(“cooled” configurations)
The chiral nature of the QCD vacuum

\[ \tan\left(\frac{\pi}{4}(1 + X(x)) \right) = \frac{|\psi_L(x)|}{|\psi_R(x)|} = \left( \frac{\psi_R^\dagger(x)\psi_L(x)}{\psi_R^\dagger(x)\psi_R(x)} \right)^{1/2} \]

The chiral nature of the QCD vacuum

Chirality Correlation within Dirac Eigenvectors from Domain Wall Fermions

T. Blum$^a$, N. Christ$^b$, C. Cristian$^b$, C. Dawson$^c$, X. Liao$^b$, G. Liu$^b$, R. Mawhinney$^b$, L. Wu$^b$, Y. Zhestkov$^b$


Consistent with the instanton picture of the vacuum
Callan-Dashen-Gross; Shuryak; Diakonov; ...

we find a striking correlation between the magnitude of the chirality density, $|\psi(x)^\dagger \gamma^5 \psi(x)|$, and the normal density, $\psi(x)^\dagger \psi(x)$, for the low-lying Dirac eigenvectors.
Sphaleron transitions at finite energy or temperature

\[ \Gamma = \frac{1}{2} \lim_{t \to \infty} \lim_{V \to \infty} \int_0^t \langle (q(x)q(0) + q(0)q(x)) \rangle d^4x \]

Sphalerons: random walk of topological charge at finite T:

\[ \langle Q^2 \rangle = 2\Gamma V t, \quad t \to \infty. \]
Topological number diffusion at strong coupling

Chern-Simons number diffusion rate at strong coupling

\[ \Gamma = \frac{(g_{YM}^2N)^2}{256\pi^3} T^4 \]

NB: This calculation is completely analogous to the calculation of shear viscosity that led to the "perfect liquid"

D. Son, A. Starinets
hep-th/020505
Classical topological solutions at strong coupling?

yes: D-instantons in (dual) weakly coupled supergravity

D-instanton as an Einstein-Rosen wormhole; the flow of RR charge down the throat of the wormhole describes change of chirality


D-instantons as a source of multiparticle production in N=4 SYM?

DK, E. Levin, arXiv:0910.3355
Instantons and multiparticle production in QCD


Reasonable phenomenologically, but strong sensitivity to the instanton size cutoff

Also: double diffractive production
E. Shuryak, E. Zahed ‘01
Instantons and the color glass condensate


Large-size instantons are suppressed:

\[ n_{sat}^{AA}(\rho) = n_0(\rho) \exp \left( -\frac{c \rho^4 Q_s^4}{8 \alpha_s^2 N_c (Q_s \tau_0)^2} \right) \]

instanton calculus becomes well-defined in the “glasma” environment
Chern-Simons diffusion
and the color glass condensate

Longitudinal “glasma” fields with a topological contents

Charge asymmetry w.r.t. reaction plane as a signature of topological fluctuations


Electric dipole moment of QCD matter!
From QCD back to electrodynamics: Maxwell-Chern-Simons theory

\[ \mathcal{L}_{\text{MCS}} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - A_\mu J^\mu + \frac{c}{4} P_\mu J_{CS}^\mu \]

\[ J_{CS}^\mu = \epsilon^{\mu \nu \rho \sigma} A_\nu F_{\rho \sigma} \quad P_\mu = \partial_\mu \theta = (\dot{\theta}, \vec{P}) \]

\[ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} + c \left( \dot{\theta} \vec{B} - \vec{P} \times \vec{E} \right), \]

\[ \nabla \cdot \vec{E} = \rho + c \vec{P} \cdot \vec{B}, \]

\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \]

\[ \nabla \cdot \vec{B} = 0, \]

EM fields in QCD “aether”
The Chiral Magnetic Effect I:
Charge separation

\[ \nabla \cdot \vec{E} = \rho + c \vec{P} \cdot \vec{B} \]

\[ \vec{P} \equiv \nabla \theta \]

\[ d_e = \sum_f q_f^2 \left( e \frac{\theta}{\pi} \right) \left( \frac{eB \cdot S}{2\pi} \right) L \]

DK '04;
DK, A. Zhitnitsky '06
The chiral magnetic effect II: chiral induction

\[ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} + c \left( \dot{\theta} \vec{B} - \vec{P} \times \vec{E} \right) \]

\[ \vec{J} = -\frac{e^2}{2\pi^2} \dot{\theta} \vec{B} \]

DK, L. McLerran, H. Warringa ’07;
K. Fukushima, DK, H. Warringa ’08;
DK, H.Warringa arXiv:0907.5007
Holographic chiral magnetic effect: the strong coupling regime (AdS/CFT)

\[ j = \sigma \chi B \]

Strong coupling

Weak coupling

H.-U. Yee, arXiv:0908.4189,
JHEP 0911:085, 2009;
V. Rubakov, arXiv:1005.1888, ...

D.K., H. Warringa
Phys Rev D80 (2009) 034028

A. Rebhan et al, JHEP 0905, 084 (2009), G.Lifshytz, M.Lippert, arXiv:0904.4772; ...

Chiral separation: D. Son and P. Surowka, ‘09
CME in the chirally broken phase
G. Basar, G. Dunne, DK, arXiv: 1003.3464; PRL

“Chiral spiral” in (1+1) theories: V. Schoen, M. Thies, hep-th/0008175

Gross-Neveu:

\[ \mathcal{L} = \bar{q} i \gamma^\mu \partial_\mu q + \frac{1}{2} g^2 \left[ (\bar{q} q)^2 - \lambda (\bar{q} \gamma^5 q)^2 \right] - m_0 \bar{q} q \]

‘t Hooft:

\[ \mathcal{L} = \bar{q} i \slashed{D} q - \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu}, \quad \slashed{D} = \gamma^\mu (\partial_\mu + ig A_\mu) \]

because of constraints on Dirac matrices in 1+1, explicit form e.g.
\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

there is an intricate connection between the vector (baryon) and chiral currents:

\[ j^0_V = j^1_A, \quad j^1_V = j^0_A \]

Baryon density - chiral current; chiral density - vector current

Son et al
Plane waves describing the pairing fermions acquire a phase difference due to the chemical potential - the spiral nature of condensates.

Gapless collective spiral excitation that carries a vector current (at finite chirality) or a chiral current (at finite baryon density).

\[
\langle J^3 \rangle = \frac{eB}{2\pi} \frac{e\mu_5}{\pi} \quad \langle J^3_5 \rangle = \frac{eB}{2\pi} \frac{e\mu}{\pi}
\]

\[4 = 2 \times (1+1)\]

\[
\langle J^1 \rangle = C^2 \cos(2\mu_5 z - \phi_R) - D^2 \cos(2\mu_5 z + \phi_L)
\]

\[
\langle J^2 \rangle = -C^2 \sin(2\mu_5 z - \phi_R) + D^2 \sin(2\mu_5 z + \phi_L)
\]

\[
\langle J^1_5 \rangle = C^2 \cos(2\mu_5 z - \phi_R) + D^2 \cos(2\mu_5 z + \phi_L)
\]

\[
\langle J^2_5 \rangle = -C^2 \sin(2\mu_5 z - \phi_R) - D^2 \sin(2\mu_5 z + \phi_L)
\]
“Quark-gluon solenoid”,
*Physics*,
June 18, 2010
“Numerical evidence for chiral magnetic effect in lattice gauge theory”,
P. Buividovich, M. Chernodub, E. Luschevskaya, M. Polikarpov, ArXiv 0907.0494; PRD

Red - positive charge
Blue - negative charge

SU(2) quenched, Q = 3; Electric charge density (H) - Electric charge density (H=0)
"Chiral magnetic effect in 2+1 flavor QCD+QED",
M. Abramczyk, T. Blum, G. Petropoulos, R. Zhou, ArXiv 0911.1348;
Columbia--RIKEN--BNL--Bielefeld

2+1 flavor Domain Wall Fermions, fixed topological sectors, 16^3 x 8 lattice

Red - positive charge
Blue - negative charge
Effective particle distribution for a certain $Q$.

\[
\frac{dN_\alpha}{d\phi} \propto 1 + 2v_{1,\alpha} \cos(\Delta\phi) + 2v_{2,\alpha} \cos(2\Delta\phi) + \ldots + 2a_{1,\alpha} \sin(\Delta\phi) + 2a_{2,\alpha} \sin(2\Delta\phi) + \ldots,
\]

\[
\Delta\phi = (\phi - \Psi_{RP})
\]

- The effect is too small to observe in a single event
- The sign of $Q$ varies and $\langle a \rangle = 0$ (we consider only the leading, first harmonic) → one has to measure correlations, $\langle a_\alpha a_\beta \rangle$, $P$-even quantity (!)
- $\langle a_\alpha a_\beta \rangle$ is expected to be $\approx 10^{-4}$
- $\langle a_\alpha a_\beta \rangle$ cannot be measured as $\langle \sin \varphi_\alpha \sin \varphi_\beta \rangle$ due to large contribution from effects not related to the orientation of the reaction plane → study the difference in corr’s in- and out-of-plane

A practical approach: three particle correlations:

\[
\langle \cos(\phi_\alpha + \phi_\beta - 2\phi_c) \rangle = \langle \cos(\phi_\alpha + \phi_\beta - 2\Psi_{RP}) \rangle v_{2,c}
\]

\[
\langle \cos(\phi_\alpha + \phi_\beta - 2\Phi_{RP}) \rangle = \langle \cos(\Delta\phi_\alpha \cos(\Delta\phi_\beta) - \sin(\Delta\phi_\alpha \sin(\Delta\phi_\beta))
\]

\[
= [\langle v_{1,\alpha}v_{1,\beta} \rangle + B^{in}] - [\langle a_{\alpha}a_{\beta} \rangle + B^{out}].
\]

\[
B^{in} \approx B^{out}, \ v_{1} = 0
\]
Azimuthal Charged-Particle Correlations and Possible Local Strong Parity Violation

NB: P-even quantity (strength of P-odd fluctuations)
Relatively good agreement between PHENIX & STAR
Are the observed fluctuations of charge asymmetries a convincing evidence for the local parity violation?

A number of open questions that still have to be clarified:

in-plane vs out-of-plane, new observables?  
\textit{e.g.} A. Bzdak, V. Koch, J. Liao,  
arXiv:0912.5050; 1005.5380; ...

physics “backgrounds”  
\textit{e.g.} M. Asakawa, A. Majumder, B. Muller,  
arXiv:1003.2436  
S. Pratt and S. Schlichting, arXiv:1005.5341  
F. Wang, arXiv: 0911.1482; ...

Fortunately, a number of analytical and numerical (lattice) tools are available to theorists, and the new data (low energy, PID asymmetries, U-U) will hopefully come - this question can be answered!
Topology, chirality, and the fragmentation of polarized quarks

Z. Kang, DK, arXiv:1006.2132

\[ D_{\pi/q^\uparrow}(z, p_\perp) = D(z, p_\perp^2) + H_1(z, p_\perp^2) \left( \frac{\vec{k} \times p_\perp}{M} \cdot s_q \right) + \tilde{H}_1(z, p_\perp^2) \frac{p_\perp \cdot s_q}{M} \]
Topology, chirality, and the fragmentation of polarized quarks

Z. Kang, DK, arXiv:1006.2132

Quark fragmentation function:

\[ \Delta(z, p_\perp) = \frac{1}{z} \int \frac{dy_- d^2 y_\perp}{(2\pi)^3} e^{ik \cdot y_-} \langle 0 | \mathcal{L}_y \psi(y) | PX \rangle \]
\[ \langle PX | \bar{\psi}(0) \mathcal{L}_0^\dagger | 0 \rangle |_{y^+ = 0}, \]

Equation of motion for the quark field:

\[ (i \gamma^\mu - m + \bar{\theta} \gamma^5) \psi(x) = 0 \]

Interactions with topological vacuum fluctuations induces local violation of Parity;
Relaxing P conservation, after twist expansion:

\[ \Delta(z, p_\perp) = \frac{1}{2} \left[ D(z, p_\perp^2) \bar{\mathbf{k}} + H_1^\dagger(z, p_\perp^2) \sigma^{\mu\nu} \frac{p_{\perp \mu} \bar{m}_\nu}{M} \right] \]
\[ + \frac{1}{2} \left[ \tilde{D}(z, p_\perp^2) \bar{\mathbf{k}} \gamma^5 \right. \]
\[ \left. + \tilde{H}_1^\dagger(z, p_\perp^2) \sigma^{\mu\nu} i \gamma^5 \frac{p_{\perp \mu} \bar{m}_\nu}{M} \right], \]

\[ \bar{\theta}_\mu \equiv \partial_\mu \theta / 2N_f \]
Topology, chirality, and the fragmentation of polarized quarks

Z. Kang, DK, arXiv:1006.2132

Quark propagator:

\[ i\tilde{S}(p, \bar{\theta}) = i/(p - m + \bar{\theta}\gamma^5) \]

in terms of the free one and L- and R-projectors:

\[
i\tilde{S}(p, \bar{\theta}) = i \left[ \mathcal{P}_R S(p + \bar{\theta}) + \mathcal{P}_L S(p - \bar{\theta}) \right] \\
\times \left[ 1 + m\gamma^5 \left( S(p + \bar{\theta}) - S(p - \bar{\theta}) \right) \right] \\
\times \left[ 1 + \frac{4m^2\bar{\theta}^2}{((p + \theta)^2 - m^2)((p - \theta)^2 - m^2)} \right]^{-1} \tag{9} \]

Evaluating the fragmentation function:

\[
\langle 0 | \psi(y) |PX \rangle \propto \langle 0 | \bar{\psi}(y) \gamma^\mu \psi(x) \gamma^5 \partial_\mu \pi(x) \bar{\psi}(x) |q(k - p), \pi(p) \rangle \\
\propto i\tilde{S}(k, \bar{\theta}) p \gamma^5 \bar{u}(k - p),
\]
Cross section in $e^+e^-$ annihilation: P-odd x P-odd is allowed!

\[
\frac{d\sigma}{dz_1dz_2d\cos\theta d(\phi_1 + \phi_2)} = \sigma_0 \sum_q e_q^2 \left\{ (1 + \cos^2 \theta) \left[ D_q(z_1)D_{\bar{q}}(z_2) - \tilde{D}_q(z_1)\tilde{D}_{\bar{q}}(z_2) \right] \\
+ \sin^2 \theta \cos(\phi_1 + \phi_2) \left[ H_q(z_1)H_{\bar{q}}(z_2) + \tilde{H}_q(z_1)\tilde{H}_{\bar{q}}(z_2) \right] \\
+ \sin^2 \theta \sin(\phi_1 + \phi_2) \left[ \tilde{H}_q(z_1)\tilde{H}_{\bar{q}}(z_2) - \tilde{H}_q(z_1)\tilde{H}_{\bar{q}}(z_2) \right] \right\} \]

"Collins effect"

P-odd, only EbyE

Physical picture:

- P-odd times P-odd terms: Data and tests (Belle, RHIC, JLab, EIC...) forthcoming

- P-odd term alone:
Could the Collins effect be a manifestation of QCD topology?

\[ I(\tilde{\theta}, z_1, z_2) = \frac{H_q^+(z_1) \tilde{H}_q^+(z_2)}{D_q(z_1) D_q(z_2) - \tilde{D}_q(z_1) \tilde{D}_q(z_2)} \sim 1.5\% \]

\[ z_1 = z_2 = 0.5 \]

\[ \bar{\theta}_\perp \sim \frac{1}{2N_f} \bar{\theta}_\perp (\bar{x}, t) \sim \frac{1}{2N_f} \cdot \frac{1}{\rho} \cdot \frac{\rho^2}{R^2} \sim 10 \text{ MeV} \]

\begin{align*}
\tilde{D}(z, p_{\perp}^2) &= \frac{g_A^2}{64 f_\pi^2 \pi^3 z} \frac{4 \bar{\theta}_\perp p_{\perp}^2}{z^2 m_q^2 + (1 - z) m_\pi^2} \left[ 1 - \frac{z}{2} \right] \\
&\quad - \frac{4(1 - z)^2 z^2 m_q^2 m_\pi^2}{(p_{\perp}^2 + z^2 m_q^2 + (1 - z) m_\pi^2)^2}, \quad (11) \\
\tilde{H}_q^+(z, p_{\perp}^2) &= \frac{g_A^2 m_q m_\pi}{4 f_\pi^2} \frac{\bar{\theta}_\perp}{8 \pi^3} \left( z - 2 \right)^3 \\
&\times \left[ (3z - 2) m_\pi^2 - 4(p_{\perp}^2 - z^2 m_q^2) \right], \quad (12)
\end{align*}

Forthcoming results for SIDIS; re-analysis of the data including (P-even) products of P-odd terms is needed.

Can be studied at EIC!
Summary

• Topological solutions are an indispensable part of non-Abelian gauge theories

• The role of topology in the description of hard processes, especially those involving polarization, must be revisited

• Topology may hold the key to the understanding of many puzzling observations in spin physics

• New measurements are needed - Belle, RHIC, JLab, EIC