Lattice QCD Thermodynamics at zero and nonzero baryon density

Christian Schmidt

for RBC-Bielefeld-GSI Collaboration

based on:

for HotQCD Collaboration

based on preliminary data, see also:
R. Soltz, INT-10-2a, June 17, INT Seattle, WA, USA.
W. Söldner, Lattice 2010, Villasimius, Sardinia, Italy.
A. Bazavov, Lattice 2010, Villasimius, Sardinia, Italy.
Analyzing the equation of state in the $(T, \mu_B)$-plane from first principles

→ determination of the phase diagram

→ understanding underling mechanism of the transition
Overview:

★ Lattice QCD at high temperature:
  • getting lattice errors under control
  • analyzing the critical behavior

The lattice tasks
Overview:

★ Lattice QCD at high temperature:
  • getting lattice errors under control
  • analyzing the critical behavior

★ Lattice QCD at high temperature and nonzero density
  • hadronic fluctuations and correlations
  • curvature of the critical line
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★ Lattice QCD at high temperature:
  • getting lattice errors under control
  • analyzing the critical behavior

★ Lattice QCD at high temperature and nonzero density
  • hadronic fluctuations and correlations
  • curvature of the critical line
  • update on the critical point determination
QCD on the lattice

\[
U_\mu(x) = P \exp \left\{ ig \int_x^{x+\hat{\mu}a} dx_\mu A_\mu(x) \right\}
\]

lattice spacing \(a\)

- discretize space time and hence all „paths“ of quarks and gluons

- lattice spacing: \(a\)
  - continuum limit: \(a \to 0\)
  - momentum cutoff \(\mathcal{O}(1/a)\)
  - observables in units of \(a\)

- freedom of choosing the lattice action
  (QCD has to be recovered in the continuum limit)

- different lattice groups mainly differ by their choice of the lattice action
Two different improvements

- Improving the dispersion relation
- Improving the flavor symmetry breaking

- Discretization of covariant derivative
  - std
  - Naik
  - p4

- Remove $O(a^2)$-effects
  - Remove the high frequency modes
    - fat3
    - fat7

- Multi-level smearing, where links remain in SU(3)

- Important to obtain the correct Stefan-Boltzmann limit
- Important to obtain the correct hadronic spectrum
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RBC-Bielefeld: p4fat3-action

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\[ \text{remove } O(a^2) \text{-effects} \]

- discretization of covariant derivative
- remove the high frequency modes

**MILC:**
- asqtad-action
- multi-level smearing, where links remain in SU(3)

**MILC:**
- Naik
- std
- fat7
- p4

\[ \text{important to obtain the correct Stefan-Boltzmann limit} \]
\[ \text{important to obtain the correct hadronic spectrum} \]
Two different improvements

- improving the dispersion relation
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→ discretization of covariant derivative

- std
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Budapest-Wuppertal: stout-action

→ removing the high frequency modes

- fat3
- 3-staple

→ important to obtain the correct Stefan-Boltzmann limit

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Two different improvements

- improving the dispersion relation
- removing $O(a^2)$-effects
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→ discretization of covariant derivative

- std
- Naik
- p4

MILC/HotQCD
future plans: HISQ-action

- fat3
- fat7

→ removing the high frequency modes

→ multi-level smearing, where links remain in SU(3)

→ important to obtain the correct Stefan-Boltzmann limit

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Two different improvements

improving the dispersion relation

\(\rightarrow\) discretization of covariant derivative (free case)

\[ \frac{p}{p_{SB}} \]

\(\mu/T=0.0, m/T=0.0\)
\(\mu/T=1.0, m/T=0.0\)
\(\mu/T=0.0, m/T=1.0\)
\(\mu/T=1.0, m/T=1.0\)

standard action

\(\frac{p}{p_{SB}} = 1 + 0 \)  \( \mu/T=1.0, m/T=1.0 \)

p4 action

\[ \frac{p}{p_{SB}} = 1 + \frac{248}{147} \left( \frac{\pi}{N_T} \right)^2 + \frac{635}{147} \left( \frac{\pi}{N_T} \right)^4 + \ldots \]  \(\text{(standard)}\)

\[ \frac{p}{p_{SB}} = 1 + 0 - \frac{1143}{980} \left( \frac{\pi}{N_T} \right)^4 + \frac{73}{2079} \left( \frac{\pi}{N_T} \right)^6 + \ldots \]  \(p4\)

\[ \frac{p}{p_{SB}} = 1 + 0 - \frac{1143}{980} \left( \frac{\pi}{N_T} \right)^4 - \frac{365}{77} \left( \frac{\pi}{N_T} \right)^6 + \ldots \]  \(Naik\)

[\(\text{P. Hegde et al., EPJC 55 (2008) 423.}\)]

\(\rightarrow\) small mass and chemical potential dependence
Two different improvements

- remove $O(a^2)$-effects
- improving the flavor symmetry breaking
- remove the high frequency modes
- pseudo-scalar mesons most affected by flavor symmetry breaking
- splittings are to good approximation mass independent (shown for HISQ: $m_l/m_s = 0.2$)

[A. Bazavov, Lattice 2010, Villasimus, Sardinia, Italy]
Two different improvements

- remove $O(a^2)$-effects
- improving the flavor symmetry breaking
  - remove the high frequency modes
  - pseudo-scalar mesons most affected by flavor symmetry breaking
  - splittings are to good approximation mass independent (shown for HISQ: $m_l/m_s = 0.2$)
  - cutoff effects in other hadronic channels are also significantly reduced

[A. Bazavov, Lattice 2010, Villasimus, Sardinia, Italy]
hadron resonance gas

\[
\ln Z(T, V, \mu_B, \mu_S, \mu_Q) = \sum_{i \in \text{hadrons}} \ln Z_{m_i}(T, V, \mu_B, \mu_S, \mu_Q) + \sum_{i \in \text{mesons}} \ln Z_{m_i}^B(T, V, \mu_S, \mu_Q) + \sum_{i \in \text{baryons}} \ln Z_{m_i}^F(T, V, \mu_B, \mu_S, \mu_Q)
\]

mesons:
\[
\frac{p_i}{T^4} = \frac{d_i}{\pi^2} \left( \frac{m_i}{T} \right)^2 \sum_{l=1}^{\infty} (1)^{l+1} K_2(lm_i/T) \cosh(lS_i\mu_S/T + lQ_i\mu_Q/T)
\]

baryons:
\[
\frac{p_i}{T^4} = \frac{d_i}{\pi^2} \left( \frac{m_i}{T} \right)^2 \sum_{l=1}^{\infty} (-1)^{l+1} K_2(lm_i/T) \cosh(lB_i\mu_B/T + lS_i\mu_S/T + lQ_i\mu_Q/T)
\]

universal scaling
\[
b^d f_s(t, h, \ldots) = f_s(b^{yt} t, b^{yh} h, \ldots)
\]

perturbation theory (\(\mathcal{O}(g^6[\ln(1/g) + \text{const.}])\))

free quark gas (\(\mathcal{O}(g^0)\))

\[
\frac{p}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,\ldots} \left[ \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu_f}{T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_f}{T} \right)^4 \right]
\]
Simulation parameters

- $N_f=2+1$: two degenerate u/d quarks + strange quark
- RHMC algorithm
- Lines of constant physics:
  - $m_l/m_s = 0.1$
  - $m_l/m_s = 0.05$
  - $m_\pi \approx 220$ MeV
  - $m_\pi \approx 150$ MeV
- Lattice size:
  - $N_\sigma/N_\tau = 4$, $N_\tau = 4, 6, 8, 12$
  - $T = \frac{1}{N_\tau a}$
  - $a = 0.25, 0.17, 0.13, 0.08$ fm
  (at $T = 200$ MeV)
- Actions:
  - p4
    - $N_\tau = 4, 6, 8$
    - $m_l/m_s = \ldots, 0.2, 0.1, 0.05$
  - asqtad
    - $N_\tau = 4, 6, 8, 12$\star
    - $m_l/m_s = \ldots, 0.2, 0.1, 0.05$
  - HISQ\star
    - $N_\tau = 6, 8$
    - $m_l/m_s = 0.2, 0.05$

\* hotQCD preliminary
The critical temperature at $\mu_B = 0$

- First piece of the puzzle:

- Disconnected chiral susceptibility is sensitive to the singular part of the free energy, divergent in the chiral limit.

\[ \chi_{m,l} (T_{m,l}) \sim m_l^{1/\delta - 1} \]

$T_{m,l}$: pseudo-critical temperature

$\delta$: critical exponent ($\sim 4.8$)

(discerned) chiral susceptibility

\[ \chi_{m,l} = \frac{T}{V} \frac{\partial^2}{\partial m_l^2} \ln Z \]

hotQCD preliminary

[W. Söldner, Lattice 2010, Villasimus, Sardinia, Italy]
The critical temperature at $\mu_B = 0$

- first piece of the puzzle:

- disconnected chiral susceptibility is sensitive to the singular part of the free energy, divergent in the chiral limit

- peak positions of asqtad $N_\tau = 12$ and HISQ $N_\tau = 8$ agree

(disconnected) chiral susceptibility

$$\chi_{m,l} = \frac{T}{V} \frac{\partial^2}{\partial m_l^2} \ln Z$$

hotQCD preliminary

[W. Söldner, Lattice 2010, Villasimus, Sardinia, Italy]
The critical temperature at $\mu_B = 0$

- first piece of the puzzle:

$$T_{m,l} = a + b \left( \frac{m_l}{m_s} \right)^{1/\beta \delta} + \frac{c}{N^2}$$

- combined chiral and continuum extrapolation with 3 fit-parameter (a,b,c)

- continuum extrapolated pseudo-critical temperature at physical masses $\frac{m_l}{m_s} = \frac{1}{27}$

$$T_{m,l} = 164 \pm 6 \text{ MeV (stat. and syst.)}$$

[hotQCD preliminary]

[W. Söldner, Lattice 2010, Villasimus, Sardinia, Italy]
• Trace anomaly: \( \epsilon - 3p \) directly accessible on the lattice
  \( \rightarrow \) use standard thermodynamic relations to extract EoS

\[ (\epsilon - 3p)/T^4 \]

\[ (\epsilon - 3p)/T^4 \]

\( T > 300 \text{ MeV} \), more works needs to be done for low temperatures

\( \rightarrow \) smooth parametrizations available to use for hydro, see also
[R. Soltz, INT 10-2a, June 17] and [P. Huovinen, INT 10-2a, June 11]
Critical Behavior of QCD (I)

- Chiral symmetry of 2-flavor QCD: $SU_L(2) \times SU_R(2) \simeq O(4)$

- Hence, if $m_s$ is large in (2+1)-flavor QCD:
  
  Expect universal behavior as of 3d-$O(4)$ spins in the vicinity of $T_c$ and the chiral limit.

- So far no clear evidence from simulations.

- Staggered fermions preserve a flavor non-diagonal $U(1)$-part of chiral symmetry even at $\alpha > 0$.
  
  → Look for $O(2)$-critical behavior.

Simulations with improved staggered fermions (p4fat3)

<table>
<thead>
<tr>
<th>$m_l/m_s$</th>
<th>$m_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/80</td>
<td>75 MeV</td>
</tr>
<tr>
<td>1/40</td>
<td>105 MeV</td>
</tr>
<tr>
<td>1/20</td>
<td>150 MeV</td>
</tr>
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</table>

Crossover at physical point: [Aoki at al., Nature 443:675-678, 2006.]
The scaling hypothesis

• Thermodynamics in the vicinity of a critical point:

\[
- \frac{1}{V} \ln Z = f_s(t, h) + f_r(T, V, H)
\]

(singular part) \hspace{1cm} \text{(regular part)}

where

\[
t = \frac{1}{t_0} \frac{T - T_c}{T_c}
\]

(reduced temperature)

\[
h = \frac{H}{h_0}
\]

(external field)

QCD:

\[
H \sim m_q
\]

(quark mass)

our choice:

\[
H = \frac{m_l}{m_s}
\]

assume:

\[
f_s(t, h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h)
\]

choose: \( b = h^{-1/y_h} \)

\[
f_s(t, h) = h_0 h^{1+1/\delta} f_M(z) \quad \text{with} \quad z = t/h^{1/\beta \delta}
\]

(“magnetic version” of the fee energy density) (scaling variable)
Magnetic EoS in $O(N)$-spin models

- order parameter (magnetization):
  $$M = -\frac{\partial f_s(t, h)}{\partial H} = \frac{1}{h_0} \frac{\partial f_s(t, h)}{\partial h} \equiv h^{1/\delta} f_G(z)$$

- universal scaling function
  $$f_G(z) = - \left[ \left(1 + \frac{1}{\delta}\right) f_M(z) - \frac{z}{\beta\delta} f'_M(z) \right]$$

- scaling variable:
  $$z = t/h^{1/\beta\delta}$$

- scaling function and critical exponents are known to high precision in condensed matter literature [e.g. Engels et al.]

- scaling function includes Goldstone effect in the limit of $z \to -\infty$
  $$z \to -\infty : \quad h \to 0, \ t < 0 \quad M \sim (-t)^\beta + c(t)\sqrt{h}$$
Magnetic EoS in QCD (Nt=4)

- two order parameter:
  \[ M_0 = m_s \langle \bar{\psi} \psi \rangle_l / T^4 \]
  \[ M = m_s \left( \langle \bar{\psi} \psi \rangle_l - m_l \langle \bar{\psi} \psi \rangle_s \right) / T^4 \]
  \[ = h^{1/\delta} f_G(z) \]

(subtracted condensate to remove UV-div. \( \sim m_l / a^2 \))

- three fit parameter: critical temperature \( T_c \) (critical coupling \( \beta_c \)), normalization constants \( t_0, h_0 \)

\[ \langle \bar{\psi} \psi \rangle_l - m_l \langle \bar{\psi} \psi \rangle_s \]

\( M_0 / h^{1/\delta} \) and \( M / h^{1/\delta} \) graphs with parameters:

- \( \beta_c = 3.2965(6) \)
- \( h_0 = 0.0022(2) \)
- \( t_0 = 0.0038(1) \)
- \( z_0 = 6.6(4) \)

- \( \beta_c = 3.2979(7) \)
- \( h_0 = 0.0035(4) \)
- \( t_0 = 0.0044(1) \)
- \( z_0 = 7.6(5) \)

\[ O(2) f_G(z) \]
Magnetic EoS in QCD (Nt=4)

• two order parameter:

\[
M_0 = m_s \langle \bar{\psi} \psi \rangle_l / T^4
\]

\[
M = m_s \left( \langle \bar{\psi} \psi \rangle_l - \frac{m_l}{m_s} \langle \bar{\psi} \psi \rangle_s \right) / T^4 \right) = h^{1/\delta} f_G(z)
\]

(subtracted condensate to remove UV-div. $\sim m_l / a^2$)

• three fit parameter: critical temperature $T_c$ (critical coupling $\beta_c$), normalization constants $t_0, h_0$

![Graphs showing $M_0$ and $M$ as functions of $T/T_c$ for different $m_l/m_s$ ratios.](image-url)
Magnetic EoS in QCD (Nt=4)

- O(2) slightly preferred, however, re-parametrization \( z \rightarrow 1.2z \) moves O(2) onto O(4)

\[ \rightarrow \text{scaling functions almost indistinguishable, we can not discriminate between O(2) and O(4)} \]

- \( z_0 = t_0/h_0^{1/\beta \delta} \) is independent under re-scaling ( \( t_0, h_0 \) not)

- \( z_0(m_s, a^2) \) might be a QCD invariant, which only depend on strange quark mass and lattice artifacts

\[ \beta_c = 3.2965(6) \]
\[ h_0 = 0.0022(2) \]
\[ t_0 = 0.0038(1) \]
\[ z_0 = 6.6(4) \]

\[ \beta_c = 3.2979(7) \]
\[ h_0 = 0.0035(4) \]
\[ t_0 = 0.0044(1) \]
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Deviations from scaling (Nt=4)

- mass range $m_l/m_s < 1/20$ is well described by scaling function
- deviations from scaling substantial for $m_l/m_s > 1/20$
- include regular part into the fit:

$$M = h^{1/\delta} f_G(z) + a_t(T - T_c)H + b_1H + b_3H^3$$

→ results for $\beta_c, t_0, h_0$ are recovered within errors
Magnetic EoS in QCD (Nt=8)

- \( N_\tau = 8 \): fit w/o scaling violations not possible yet
- fit for \( \beta_c, t_h, h_0, a_t, h_1, h_3 \) (range \( m_l/m_s \geq 1/40 \)) works reasonably well → assume \( z_0 \) to be stable/reliable
- cutoff dependence:

\[
\begin{array}{c|cc}
N_\tau & 4 & 8 \\
\hline
z_0 & 7.5(9) & 4.3(5) \\
\end{array}
\]

→ further studies are needed to control continuum limit
Magnetic EoS in QCD (Nt=8)

- $N_\tau = 8$: fit w/o scaling violations not possible yet
- fit for $\beta_c, t_h, h_0, a_t, h_1, h_3$ (range $m_l/m_s \geq 1/40$) works reasonably well → assume $z_0$ to be stable/reliable
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→ further studies are needed to control continuum limit

\[
(M_0 - a_t^0 t H - b_1 H - b_3 H^3)/h^{1/6}\]

\[
\beta_c = 3.2981(7)
\]

\[
h_0 = 0.0036(4)
\]

\[
t_0 = 0.0041(1)
\]

\[
z_0 = 8.4(7)
\]

\[
b_1 = 3.0(4)
\]

\[
a_t = 9.1(20)
\]

\[
b_3 = -11.5(49)
\]

\[
N_\tau = 8: m_l/m_s
\]

\[
1/5
\]

\[
1/10
\]

\[
1/20
\]

\[
1/40
\]

\[
1/80
\]

\[
(M_0 - a_t^0 t H - b_1 H - b_3 H^3)/h^{1/6}\]

\[
\beta_c = 3.5018(0)
\]

\[
h_0 = 0.0006(0)
\]

\[
t_0 = 0.0029(1)
\]

\[
z_0 = 4.1(2)
\]

\[
b_1 = 2.9(5)
\]

\[
a_t = -8.2(16)
\]

\[
b_3 = 15.4(73)
\]

\[
N_\tau = 8: m_l/m_s
\]

\[
1/5
\]

\[
1/10
\]

\[
1/20
\]

\[
1/40
\]
Critical Behavior of QCD (II)

- situation at nonzero chemical potential is very unclear
- direct simulations MC simulations are prohibited by the sign-problem

→ use Taylor expansion approach

expected \((T, \mu)\)-phase diagrams:

\[
\begin{align*}
T & \quad m_u = m_d = 0 \\
& \quad m_s > m^\text{tri}_s \\
& \quad \text{line of 2. order transitions } O(4)
\end{align*}
\]

\[
\begin{align*}
T & \quad m_u = m_d > 0 \\
& \quad m_s > m^\text{tri}_s \\
& \quad \text{line of 1. order transitions}
\end{align*}
\]
Lattice QCD at nonzero density

- direct MC-simulations for $\mu > 0$ not possible

\[
Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\{S_F(A, \psi, \bar{\psi}) - \beta S_G(A)\}
\]

\[
= \int \mathcal{D}A \det[M](A, \mu) \exp\{ -\beta S_G(A)\}
\]

complex for $\mu > 0$ Interpretation as probability is necessary for MC-Integration

$\rightarrow$ perform a Taylor expansion around $\mu = 0$
• Taylor-expansion of the pressure
\[ \frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V, T, \mu_u, \mu_d, \mu_s) = \sum_{i,j,k} c_{i,j,k}^{u,d,s} \left( \frac{\mu_u}{T} \right)^i \left( \frac{\mu_d}{T} \right)^j \left( \frac{\mu_s}{T} \right)^k \]

• calculate Taylor coefficients at fixed temperature

• no sign-problem: all simulations at \( \mu = 0 \)
\[ c_{i,j,k}^{u,d,s} \equiv \frac{1}{i!j!k!} \frac{1}{VT^3} \frac{\partial^i \partial^j \partial^k \ln Z}{\partial \left( \frac{\mu_u}{T} \right)^i \partial \left( \frac{\mu_d}{T} \right)^j \partial \left( \frac{\mu_s}{T} \right)^k} \bigg|_{\mu_u,d,s=0} \]

• expansion coefficients reflect fluctuations of various quantum numbers

**generalized susceptibilities**

\[ 2!c_2^X = \chi_2^X = \frac{1}{VT^3} \left( \langle X^2 \rangle - \langle X \rangle^2 \right) \quad \text{quadratic fluctuations} \]

\[ 4!c_4^X = \chi_4^X = \frac{1}{VT^3} \left( \langle X^4 \rangle - 3 \langle X^2 \rangle^2 \right) \quad \text{quartic fluctuations} \]

\[ X = u, d, s, B, Q, S, \cdots \]
Taylor-expansion of the pressure

\[
\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V, T, \mu_B, \mu_Q, \mu_S) = \sum_{i,j,k} c_{i,j,k}^{B,Q,S} \left( \frac{\mu_B}{T} \right)^i \left( \frac{\mu_Q}{T} \right)^j \left( \frac{\mu_S}{T} \right)^k
\]

- calculate Taylor coefficients at fixed temperature

- no sign-problem: all simulations at \( \mu = 0 \)

\[c_{i,j,k}^{u,d,s} \equiv \frac{1}{i! j! k!} \frac{1}{VT^3} \left. \frac{\partial^i \partial^j \partial^k \ln Z}{\partial \left( \frac{\mu_u}{T} \right)^i \partial \left( \frac{\mu_d}{T} \right)^j \partial \left( \frac{\mu_s}{T} \right)^k} \right|_{\mu_{u,d,s}=0}
\]

- expansion coefficients reflect fluctuations of various quantum numbers

**generalized susceptibilities**

\[2!c_2^X = \chi_2^X = \frac{1}{VT^3} \left( \langle X^2 \rangle - \langle X \rangle^2 \right) \quad \text{quadratic fluctuations}
\]

\[4!c_4^X = \chi_4^X = \frac{1}{VT^3} \left( \langle X^4 \rangle - 3 \langle X^2 \rangle^2 \right) \quad \text{quartic fluctuations}
\]

\[X = u, d, s, B, Q, S, \ldots\]
Hadronic fluctuations

at \( \mu_B > 0 \) (\( \mu_S = \mu_Q = 0 \))

baryon number fluctuations

\[
\chi_B = 2c_2^B + 12c_4^B \left( \frac{\mu_B}{T} \right)^2 + \cdots
\]

baryon number-strangeness correlations

\[
C_{BS} = \frac{c_{1,1}^{B,S} + 3c_{3,1}^{B,S} \left( \frac{\mu_B}{T} \right)^2 + \cdots}{\chi_S \left( \frac{\mu_B}{T} \right)}
\]

strangeness fluctuations

\[
\chi_S = 2c_{0,2}^{B,S} + 2c_{2,2}^{B,S} \left( \frac{\mu_B}{T} \right)^2 + \cdots
\]

\( \mu_B/T \gtrsim 0.5 \)

\( \mu_B/T \gtrsim 1.5 \)

LO introduces a peak in the fluctuations/correlations, NLO shifts the peak towards smaller temperatures

truncation errors become large at \( \mu_B/T \gtrsim 1.5 \)
Hadronic fluctuations

hadron resonance gas

$$\ln Z(T, V, \mu_B, \mu_S, \mu_Q) = \sum_{i \in \text{hadrons}} \ln Z_{m_i}(T, V, \mu_B, \mu_S, \mu_Q)$$

$$\sum_{i \in \text{mesons}} \ln Z^B_{m_i}(T, V, \mu_S, \mu_Q) + \sum_{i \in \text{baryons}} \ln Z^F_{m_i}(T, V, \mu_B, \mu_S, \mu_Q)$$

mesons:

$$\frac{p_i}{T^4} = \frac{d_i}{\pi^2} \left( \frac{m_i}{T} \right)^2 \sum_{l=1}^{\infty} (+1)^{l+1} l^{-2} K_2(lm_i/T) \cosh(lS_i\mu_S/T + lQ_i\mu_Q/T)$$

baryons:

$$\frac{p_i}{T^4} = \frac{d_i}{\pi^2} \left( \frac{m_i}{T} \right)^2 \sum_{l=1}^{\infty} (-1)^{l+1} l^{-2} K_2(lm_i/T) \cosh(lB_i\mu_B/T + lS_i\mu_S/T + lQ_i\mu_Q/T)$$

3 ratios:

$$\frac{\chi_4^B}{\chi_2^B} = \kappa \sigma^2 = \frac{B^4}{B^2} = 1$$

$$\frac{\chi_3^B}{\chi_2^B} = S \sigma = \frac{B^3}{B^2} \tanh(\mu_B/T)$$

$$\frac{\chi_2^B}{\chi_1^B} = \sigma^2/N_B = \frac{B^2}{B^1} \coth(\mu_B/T)$$
**Hadronic fluctuations**

- **Kurtosis times variance**

![Graph showing the relationship between $\frac{\chi^B_4}{\chi^B_2}$ and $T/T_c$.

Filled: $n_f=4$, $m_\pi=220$ MeV

Open: $n_f=2$, $m_\pi=770$ MeV

Resonance gas

Use parametrization of Freeze-out curve to connect to Star measurements of net-proton number

\[
T(\mu_B) = 0.166 \text{ GeV} - 0.139 \text{ GeV}^{-1} \mu_B^2 - 0.053 \text{ GeV}^{-3} \mu_B^4
\]

\[
\mu_B(\sqrt{s}) = \frac{1.308 \text{ GeV}}{1 + 0.273 \text{ GeV}^{-1} \sqrt{s}}
\]


- Sensitive to relevant quantum numbers in the medium
- Divergent at the critical point


Hadronic fluctuations

HRG vs. Experiment:

- Fluctuations increase for small $\sqrt{s}$
- Sensitive to higher order derivatives due to nearby convergence radius

Lattice vs. HRG:


- Net-proton number fluctuations can be described by the HRG

[Karsch, Redlich, arXiv:1007.2581]
method for locating of the CEP:

- determine largest temperature where all coefficients are positive \( \rightarrow T^{\text{CEP}} \)
- determine the radius of convergence at this temperature \( \rightarrow \mu^{\text{CEP}} \)

all coefficients positive: singularity on the real axis!

\[ p = c_0 + c_2 (\mu_B/T)^2 + c_4 (\mu_B/T)^4 + \cdots \]

\[ \chi_B = 2c_2 + 12c_4 (\mu_B/T)^2 + 30c_6 (\mu_B/T)^4 + \cdots \]

\[ \rho_n(p) = \sqrt{\frac{c_n}{c_{n+2}}} \]

\[ \rho = \lim_{n \to \infty} \rho_n \]
method for locating of the CEP:

- determine largest temperature where all coefficients are positive \( \rightarrow T^{CEP} \)
- determine the radius of convergence at this temperature \( \rightarrow \mu^{CEP} \)

all coefficients positive: singularity on the real axis!

first non-trivial estimate of \( T^{CEP} \) by \( c_8 \)
second non-trivial estimate of \( T^{CEP} \) by \( c_{10} \)

\[
p = c_0 + c_2 (\mu_B/T)^2 + c_4 (\mu_B/T)^4 + \cdots
\]

\[
\chi_B = 2c_2 + 12c_4 (\mu_B/T)^2 + 30c_6 (\mu_B/T)^4 + \cdots
\]

\[
\rho_n(p) = \sqrt{c_n/c_{n+2}}
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method for locating of the CEP:
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all coefficients positive: singularity on the real axis!

\[
p = c_0 + c_2 \left( \frac{\mu_B}{T} \right)^2 + c_4 \left( \frac{\mu_B}{T} \right)^4 + \cdots
\]
\[
\chi_B = 2c_2 + 12c_4 \left( \frac{\mu_B}{T} \right)^2 + 30c_6 \left( \frac{\mu_B}{T} \right)^4 + \cdots
\]

\[
\rho_n(p) = \sqrt{\frac{c_n}{c_{n+2}}}
\]
\[
\rho = \lim_{n \to \infty} \rho_n
\]
Thermal fluctuations of the order parameter

\[ \chi_t \equiv \frac{\partial M}{\partial T} = \frac{1}{t_0 T_c} \frac{\partial M}{\partial t} = \frac{1}{t_0 T_c} h^{(\beta - 1)/\beta \delta} f'_G(z) \]

where

\[ t = \frac{1}{t_0} \frac{T - T_c}{T_c} \]

(reduced temperature)

\[ h = \frac{H}{h_0} \]

(external field)

• **mixed susceptibility:**

\[ t = \frac{1}{t_0} \left( \frac{T - T_c}{T_c} + \kappa \mu \right) \]

in the chiral limit: \( \mu_l \) does not break chiral symmetry

\[ \rightarrow \text{ couples only to reduced temperature} \]

• introducing **chemical potential:**

\[ c_2^{\bar{\psi} \psi} \equiv \left. \frac{\partial^2 M}{\partial (\mu_l/T)^2} \right|_{\mu_l=0} = \frac{2\kappa \mu}{t_0 T_c} \frac{\partial M}{\partial t} = \frac{2\kappa \mu}{t_0 T_c} h^{(\beta - 1)/\beta \delta} f'_G(z) \]

\[ \propto \chi_t \]
The critical line

- curvature of critical line in the chiral limit:

\[ t = 0 \quad \rightarrow \quad \frac{T}{T_c} = 1 - \kappa_\mu \left( \frac{\mu_l}{T} \right)^2 \]

- \( t_0, h_0, T_c \) known form scaling analysis of magnetic EoS

→ fit \( 2\kappa_\mu f'_G(z) \) to \( \chi_t \)-data (one fit parameter)

→ preliminary result from fit to \( O(2) \) scaling curve:
  \[ \kappa_\mu = 0.035(1) \]

→ for orientation: reweighting std. action, \( m_l/m_s = 1/27 \)
  \[ \kappa_\mu = 0.0288(9) \]

★ HISQ action
  • distortion of spectrum drastically reduced by the HISQ action

★ Tc
  • preliminary HotQCD continuum result result:
    \[ T_{m,l} = 164 \pm 6 \text{ MeV} \text{ (stat. and syst.)} \]

★ EoS
  • no cutoff effect remain for \( T > 300 \text{MeV} \)
  • at small \( T \): HISQ and \( N_t=12 \) data closer to HRG

★ The magnetic EoS
  • EoS consistent with 3d-O(N) scaling already at physical masses
  • we find no evidence for nearby first order phase transitions
  • statistics not (yet) sufficient to discriminate between universality classes

needs to be confirmed in the continuum limit
Non-zero chemical potential

- Taylor expansion coefficients of the pressure give access to small but nonzero chemical potential

- Ratios of baryonic susceptibilities are sensitive to the critical point and rather independent of the mass spectrum

- STAR data on net-proton fluctuations can be described the hadron resonance gas

- The location of the critical point can be estimates by estimates of the radius of convergence

- The curvature of the critical line in the chiral limit can be extracted from scaling properties of mixed susceptibilities