When Perturbation Theory Fails...

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When Perturbation Theory Fails...

- SU(3) chiral perturbation theory?
- Charm quark in HQET, NRQCD?
- Extrapolations of lattice QCD data?

Solution? Not this talk...
Literal interpretation of title

Non-Perturbative Examples

- Toy model
- Hadrons in uniform electromagnetic fields
- Hyperons in SU(2) chiral perturbation theory
Toy model: \( 0 < x \ll 1 \)

\[
F(x) = \int_0^\infty \frac{e^{-s}}{1 + sx} ds
\]

Cannot series expand about 0
Toy model: \[ 0 < x \ll 1 \]

\[ F(x) = \int_0^\infty \frac{e^{-s}}{1 + sx} ds \]

Cannot series expand about 0

\[ F(x) = \int_0^\infty ds\ e^{-s} \left( \sum_{j=0}^{\infty} (-sx)^j \right) = \sum_{j=0}^{\infty} \left( \int_0^\infty ds \ s^j e^{-s} \right) (-x)^j \]

“Do it anyway”

(Physicist)

Suggests approximation

\[ F_N(x) = \sum_{j=0}^{N} \frac{(-)^j j! x^j}{j!} \]
Toy model: $0 < x \ll 1$

$$F(x) = \int_{0}^{\infty} \frac{e^{-s}}{1 + s \cdot x} ds$$

Cannot series expand about 0

$$F(x) = \int_{0}^{\infty} ds \ e^{-s} \left( \sum_{j=0}^{\infty} (-s \cdot x)^j \right) = \sum_{j=0}^{\infty} (-x)^j \left( \int_{0}^{\infty} ds \ s^j \ e^{-s} \right)$$

“Do it anyway” (Physicist)

Suggests approximation

$$F_N(x) = \sum_{j=0}^{N} (-)^j \cdot j! \cdot x^j$$

$$|F(x) - F_N(x)| = x^{N+1} \int_{0}^{\infty} \frac{s^{N+1} e^{-s} ds}{1 + s \cdot x} \leq x^{N+1} (N + 1)!$$
Toy model: $0 < x \ll 1$

$$F(x) = \int_0^\infty \frac{e^{-s}}{1 + s x} \, ds$$

$$F_N(x) = \sum_{j=0}^N (-)^j j! \, x^j$$

Minimize error for large $N$

$$|F(x) - F_N(x)| \leq x^{N+1} (N + 1)! \approx \sqrt{2\pi N} (xN)^N e^{-N}$$

“Do it anyway” (Physicist)

$$\approx \sqrt{\frac{2\pi}{x}} e^{-\frac{1}{x}}$$

$x \sim 1/N$

- Include more terms: limits to smaller $x$
- Make better for larger $x$: dropping terms

Asymptotic expansions: intuitively opposite
Toy model: \( 0 < x \ll 1 \)

\[
F(x) = \int_0^\infty \frac{e^{-s}}{1 + sx} ds
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F_N(x) = \sum_{j=0}^{N} (-)^j j! \ x^j
\]

\[
x \sim 1/N
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Toy model: \( 0 < x \ll 1 \)

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\[x \sim 1/N\]
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\[x \sim 1/N\]
Toy model: \[0 < x \ll 1\]

\[
F(x) = \int_{0}^{\infty} \frac{e^{-s}}{1 + sx} ds
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\[
F_N(x) = \sum_{j=0}^{N} (-)^j j! x^j
\]

Asymptotic expansions:
zero radius of convergence = intuitively opposite
Where have I seen this behavior?
Where have I seen this behavior?

When Perturbation Theory Fails...
Hadrons in uniform electromagnetic fields
Schwinger (1951), ..., Tiburzi (2008)

- Green’s functions exist in closed form for uniform EM fields
- Use to study non-perturbative effects

Scalar case

Magnetic field:

\[ A_\mu = (-Bx_2, 0, 0, 0) \]

Solution

\[
D(x', x) = \frac{1}{2} \int_0^\infty ds \int \frac{d\tilde{k}}{(2\pi)^3} e^{i\tilde{k} \cdot (x' - x)} \begin{pmatrix} x'_2 - \frac{k_1}{eB}, s & \bigg| & x_2 - \frac{k_1}{eB}, 0 \end{pmatrix} e^{-sE_\perp^2/2}
\]

Harmonic oscillator propagator

\[
\langle X', s | X, 0 \rangle = \sqrt{\frac{eB}{2\pi \sinh eBs}} \exp\left\{ -\frac{eB}{2\sinh eBs} \left[ (X'X^2 + X^2) \cosh eBs - 2X'X \right] \right\}
\]
Hadrons in uniform electromagnetic fields

- Chiral perturbation theory in strong QED

\[ \mathcal{L} = \frac{g^2}{8} (D_\mu \Sigma^\dagger D_\mu \Sigma) - \frac{\lambda}{2} (m_Q (\Sigma^\dagger + \Sigma)) \]

Scalar case

- Power counting

\[ \frac{k^2}{\Lambda^2_X} \sim \frac{m_\pi^2}{\Lambda^2_X} \sim \frac{eF_{\mu \nu}}{\Lambda^2_X} \sim \varepsilon^2 \]

\[ eB/m_\pi^2 \sim 1 \]

- Calculate the neutral pion energy in magnetic field
**Hadrons in uniform electromagnetic fields**

- **Chiral perturbation theory in strong QED**
- **Neutral pion energy**

\[
m_{\text{eff},\pi^0}^2 = m_{\pi}^2 \left[ 1 + \frac{2e|B|}{(4\pi f)^2} \mathcal{I}\left( \frac{m_{\pi}^2}{e|B|} \right) \right]
\]

\[eB/m_{\pi}^2 \sim 1\]

**Closed form**

\[
\mathcal{I}(x) = \int_0^\infty \frac{ds}{s^2} e^{-xs} \left( \frac{s}{\sinh s} - 1 \right) = x \left( 1 - \log \frac{x}{2} \right) + 2 \log \Gamma \left( \frac{1+x}{2} \right) - \log 2\pi
\]

- **Why? Background field lattice QCD computations in this regime**

\[
m_{\text{eff}}(B) = m - \frac{1}{2} 4\pi \beta_M B^2 + \mathcal{O}(B^4)
\]

**Closed torus can leak no flux**

\[
\exp(ieBA) \quad eB = \frac{2\pi n}{L^2}
\]
Hadrons in uniform electromagnetic fields

Chiral perturbation theory in strong QED

Neutral pion energy

\[ m_{\text{eff}, \pi^0} = m_\pi^2 \left[ 1 + \frac{2e|B|}{(4\pi f)^2} \mathcal{I} \left( \frac{m_\pi^2}{e|B|} \right) \right] \]

\[ E_{\pi^0}(p = 0) = m_\pi + \frac{m_\pi^3}{(4\pi f)^2} \left[ \frac{1}{6} \left( \frac{eB}{m_\pi^2} \right)^2 + \frac{7}{180} \left( \frac{eB}{m_\pi^2} \right)^4 - \frac{31}{630} \left( \frac{eB}{m_\pi^2} \right)^6 + \cdots \right] \]

\[ m_{\text{eff}}(B) = m - \frac{1}{2} 4\pi \beta_M B^2 + \mathcal{O}(B^4) \]

\[ \beta_M^{\pi^0} = \frac{\alpha_{\text{f.s.}}}{3(4\pi f)^2 m_\pi} \]

"Do it anyway"

(Physicist)
Hadrons in uniform electromagnetic fields

Chiral perturbation theory in strong QED

Neutral pion energy

\[ \xi = \frac{e|B|}{\sqrt{6m^2}} \]

WARNING

intuitively obvious

asymptotically opposite

⭐️ When Perturbation Theory Fails...
Hadrons in uniform electromagnetic fields

Chiral perturbation theory in strong QED

Neutral pion energy

\[ \xi = \frac{e|B|}{\sqrt{6m^2}} \]

\[ \delta E \]

When Perturbation Theory Fails...
Hadrons in uniform electromagnetic fields

- Chiral perturbation theory in strong QED

**IDLE AMUSEMENT**

- Gell-Mann / Oaks / Renner Relation
  \[ f_\pi^2 m_\pi^2 = 4\langle \bar{q}q \rangle m \]
  \[ f_\pi^2(B) m_{\pi,\text{eff}}(B) = 4\langle \bar{q}q \rangle_B m \]

- Electric Field: analytic continuation
  \[ \Gamma_{\pi^0} = -\Im(m_{\text{eff,}\pi^0}/m_\pi) \]
  Schwinger mechanism

- Charged pions

- Nucleon

  In strong magnetic fields, proton beta decays to neutron
Hyperons in SU(2) Chiral Perturbation Theory
Tiburzi and Walker-Loud (2008)
Jiang, Tiburzi, and Walker-Loud (2009)

Motivation SU(3)
Heavy baryon chiral perturbation theory

“Do it anyway” (Physicist)

\[ \frac{m_\eta}{M_B} \sim \frac{1}{2} \]

Kaon, eta contributions large & increase with strangeness

\[ m_s \sim \Lambda_{QCD} \]

SU(3) expansion precarious

\[ \delta M_N(\mu = \Lambda_X)/M_N = -39\% \]
\[ \delta M_\Lambda(\mu = \Lambda_X)/M_\Lambda = -67\% \]
\[ \delta M_\Sigma(\mu = \Lambda_X)/M_\Sigma = -89\% \]
\[ \delta M_\Xi(\mu = \Lambda_X)/M_\Xi = -98\% \]
Hyperons in SU(2) Chiral Perturbation Theory

Schematic SU(3) Expansion of Sigma Mass:

\[ M_\Sigma = M^{SU(3)} + a m^2_K + b m^3_K + \ldots \]

Large Kaon contributions

\[ m^2_K = \frac{1}{2} m^2_\pi + \frac{1}{2} m^2_\eta_s \quad m_\eta_s = 672 \text{ MeV} \]

Reorganize!

\[ \frac{m^2_\pi}{m^2_\eta_s} = 0.04 \ll 1 \]

\[ M_\Sigma = M^{SU(3)} + a' m^2_\eta_s + a'' m^2_\pi + b' m^3_\eta_s + b'' m_\eta_s m^2_\pi + b''' m^3_\pi \left( \frac{m_\pi}{m_\eta_s} \right) + \ldots \]

Expansion of Sigma Mass about the SU(2) chiral limit

\[ m_u, m_d \ll m_s \sim \Lambda_{QCD} \]
Hyperons in SU(2) Chiral Perturbation Theory

$M = M^{SU(2)} + \alpha m_{\pi}^2 + \beta m_{\pi}^3 + \beta' F(m_{\pi}, \delta)$

Trend opposite SU(3): greater strangeness, better convergence

$g_A = 1.25, \quad g_{\Sigma\Sigma} = 0.78, \quad g_{\Xi\Xi} = 0.24$

$g_{\Delta N} = 1.48, \quad g_{\Sigma^*\Sigma} = 0.76, \quad g_{\Xi^*\Xi} = 0.69$
Hyperons in SU(2) Chiral Perturbation Theory

SU(2) Perturbative Expansion can FAIL!

I) Perturbative expansion about SU(2) limit

“Duh!” (Maryland Colleague) $m_\pi/\Lambda_X$ need lattice QCD

II) Perturbative SU(2) expansion of SU(3)!

Kaon thresholds... can study non-perturbatively
Hyperons in SU(2) Chiral Perturbation Theory

SU(2) Perturbative Expansion can FAIL!
Kaon production cannot be described in SU(2)

\[ m_u, m_d \ll m_s \ll \Lambda_{QCD} \]

II) Perturbative SU(2) expansion of SU(3)!
“Exact Solution” SU(3) is theory SU(2) is asymptotically describing
Do SU(2) expansions of hyperon masses break down because of KN thresholds?

\[ m_K = 0.50 \text{ GeV} \]

\[ \delta_{N\Sigma} = 0.25 \text{ GeV} \]

\[ \delta_{N\Sigma^*} = 0.45 \text{ GeV} \]

Only virtual, but analyticities near threshold

\[ F(m_K^2, -\delta_{BB'}, \mu) = (\delta_{BB'}^2 - m_K^2)^{3/2} \log \left( \frac{-\delta_{BB'} - \sqrt{\delta_{BB'}^2 - m_K^2 + i\epsilon}}{-\delta_{BB'} + \sqrt{\delta_{BB'}^2 - m_K^2 + i\epsilon}} \right) \]
Hyperons in SU(2) Chiral Perturbation Theory

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\]

\[ m_K^2 = \frac{1}{2} m_{\pi}^2 + \frac{1}{2} m_{\eta_s}^2 \]

\[ f\left( m_K^2 - \delta_{BB'}^2 \right) = f\left( \frac{1}{2} m_{\eta_s}^2 - \delta_{BB'}^2 \right) + \frac{1}{2} m_{\pi}^2 f'\left( \frac{1}{2} m_{\eta_s}^2 - \delta_{BB'}^2 \right) + \]

"Do it anyway" (Physicist)
Hyperons in SU(2) Chiral Perturbation Theory

Do SU(2) expansions of hyperon masses break down because of KN thresholds?

\[ \mathcal{F}(m_K^2, -\delta_{BB'}) = \mathcal{F}^{(0)} + m_\pi^2 \mathcal{F}^{(2)} + m_\pi^4 \mathcal{F}^{(4)} + \ldots \]

\[ \varepsilon_{N\Sigma} = 0.05 \quad \varepsilon_{N\Sigma^*} = 0.24 \quad \varepsilon_{BB'} = 6.9 \]

\[ \delta_{N\Sigma} = 0.25 \, \text{GeV} \quad \delta_{N\Sigma^*} = 0.45 \, \text{GeV} \quad \delta = 0.485 \, \text{GeV} \]

“Do it anyway” (Physicist)

\[ \varepsilon_{BB'} = \frac{1}{2} m_\pi^2 \frac{1}{2} m_{\eta_s}^2 - \delta_{BB'}^2 \]
Hyperons in SU(2) Chiral Perturbation Theory

\[ \delta_{N\Sigma^*} = 0.45 \text{ GeV} \]

\[ \mathcal{F}(m_K^2, -\delta_{BB'}) = \mathcal{F}^{(0)} + m_\pi^2 \mathcal{F}^{(2)} + m_\pi^4 \mathcal{F}^{(4)} + \ldots \]

N=2

- Include more terms: limits to smaller pion mass
- Make better for larger pion mass: dropping terms

Asymptotic expansions: intuitively opposite
Hyperons in SU(2) Chiral Perturbation Theory

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Hyperons in SU(2)  
Chiral Perturbation Theory  

\[ \mathcal{F}(m_K^2, -\delta_{BB'}) = \mathcal{F}^{(0)} + m^2 \mathcal{F}^{(2)} + m^4 \mathcal{F}^{(4)} + \ldots \]

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Asymptotic expansions: intuitively opposite
Hyperons in SU(2)

Chiral Perturbation Theory

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Asymptotic expansions: intuitively opposite
Hyperons in SU(2)
Chiral Perturbation Theory

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\[ \delta_{N\Sigma^*} = 0.45 \text{ GeV} \]

- Include more terms: limits to smaller pion mass
- Make better for larger pion mass: dropping terms
- Asymptotic expansions: intuitively opposite
When perturbation theory fails, we probably try to use it anyway.

Asymptotic expansions: intuitively opposite

Including more terms limits to smaller range

Make better for parameters by dropping terms (limited control)

\[ eB/m_{\pi}^2 \sim 1 \]

\[ \varepsilon_{BB'} = \frac{1}{2} \frac{m_{\pi}^2}{m_{\eta_s}^2 - \delta_{BB'}^2} \]
Neutral pion in electric field: $B \rightarrow iE$