Euclidean Relativistic Quantum Mechanics

W. N. Polyzou

The University of Iowa

April 16, 2010
Collaborators

Iowa Students

Phil Kopp, Victor Wessels
Motivation

- Construct relativistic quantum models of systems with a finite number of degrees of freedom

- Desirable features:
  - Models motivated by field theory
  - Cluster properties
Elements of relativistic quantum theory

- Hilbert space $\mathcal{H}$ (requirement for a quantum theory)

- Unitary representation of the Poincaré group (requirement for relativistic invariance of quantum probabilities).

- Cluster properties (required for tests of relativity on isolated subsystems)

- Spectral condition (required for stability of theory)
Input

Reflection-positive Euclidean Green function(s) or generating functional

Problem

Construct relativistic quantum mechanical models

(We want to avoid using analytic continuation!)
Field theory motivation

Euclidean generating functional or Green functions:

\[
Z[f] := \frac{\int D_e[\phi] e^{-A[\phi]+i\phi(f)}}{\int D_e[\phi] e^{-A[\phi]}} = \sum_n \frac{(i)^n}{n!} S_n (f, \ldots, f)
\]

\(A[\phi] = \text{Action}, \ D_e[\phi] = \text{Euclidean "path measure"}\)

\(f(\tau, x) = \text{Positive Euclidean-time support test functions}\)

\(S_+ := \{ f(\tau, x) \in S | f(\tau, x) = 0, \ \tau < 0 \}\)

Euclidean time reflection

\(\theta f(\tau, x) := f(-\tau, x)\)
Reconstruction of Quantum Mechanics
Osterwalder and Schrader - C.M.P. 31(1973)83;42(1975)281

Vectors (dense set)

\[ B[\phi] = \sum_{j=1}^{N_b} b_j e^{i\phi(f_j)} \quad C[\phi] = \sum_{k=1}^{N_c} c_k e^{i\phi(g_k)} \]
\[ b_j, c_k \in \mathbb{C} \quad f_j, g_k \in \mathcal{S}_+ \quad N_b, N_c < \infty \]

Hilbert space inner product

\[ \langle B|C \rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j] \]
Remarks

$B[\phi]$  “wave functionals”

$$\langle B | C \rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j]$$

- The inner product is the physical (Minkowski) inner product!
- The generating functional and test functions are Euclidean!
- All integrals are over Euclidean space-time variables!
- No analytic continuation is used to calculate the Minkowski scalar product!
Reflection positivity

(Osterwalder-Schrader Positivity)

\[ \langle B|B \rangle \geq 0 \]

Property of \( Z[f] \) or \( \{ S_n(x_1, \cdots, x_n) \} \)

\[ M_{ij} = Z[f_i - \theta f_j] \geq 0 \quad \forall \quad \{ f_1, \cdots, f_N \} \in S_+ \]
Operator algebra

(for scattering asymptotic condition)

\[ BC[\phi] = B[\phi]C[\phi] = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j c_k e^{i\phi(g_k + f_j)} = \sum_{n=1}^{N_c} d_n e^{i\phi(h_n)} \]

\[ N_d = N_b N_c \quad h_n = g_k + f_j \quad d_n = b_j c_k \]
Cluster properties

\[ g_a(\tau, x) := g(\tau, x - a) \]

\[ \lim_{|a| \to \infty} \left( Z[f + g_a] - Z[f]Z[g] \right) \to 0 \]

\[ \lim_{|a| \to \infty} S_{m+n}(f, \ldots, f, g_a, \ldots, g_a) = S_m(f, \ldots f)S_n(g, \ldots, g) \]
Operators

\[
[T(\beta, a), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_j, \beta, a)}
\]

\[
f_{n, \beta, a}(\tau, x) := f_n(\tau - \beta, x - a) \quad \beta > 0
\]

\[
[U(R), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_j, R)}
\]

\[
f_{j, R}(\tau, x) := f_j(\tau, Rx) \quad f_j \in S_+ > 0
\]

\[
[W(\hat{n}, \psi), B][\phi] := \sum_j c_j e^{i\phi(f_k, \psi, \hat{n})}
\]

\[
f_{j, \phi, \hat{n}}(\tau, x) := f_j(\tau', x') \quad f_f \in S_{\chi,+}
\]

\[
S_{\chi,+} = \{ f \in S_+ | f(\tau, x) = 0 \quad \tan^{-1} \left( \frac{\tau}{|x|} \right) \geq \chi \}
\]

\[
\psi < \pi/2 - \chi
\]

\[
\tau' = \tau \cos(\psi) - x_{\hat{n}} \sin(\psi) \quad x'_{\hat{n}} = x_{\hat{n}} \cos(\psi) + \tau \sin(\psi)
\]
Poincaré generators

\[
[H, B][\phi] = -\frac{\partial}{\partial \beta} \left( T(\beta, 0) B \right) [\phi]_{\beta=0}
\]

\[
[P, B][\phi] = -i \frac{\partial}{\partial a} \left( T(0, a) B \right) [\phi]_{a=0}
\]

\[
[(K \cdot \hat{n}), B][\phi]) = -\frac{\partial}{\partial \psi} \left( W(\hat{n}, \psi) B \right) [\phi]_{\psi=0}
\]

\[
[(J \cdot \hat{n}), B][\phi]) = -i \frac{\partial}{\partial \psi} \left( R(\hat{n}, \psi) B \right) [\phi]_{\psi=0}
\]

\[
[M^2, B][\phi] := \left( \frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial a^2} \right) \left( T(\beta, a) B \right) [\phi]_{\beta=0, a=0}
\]
One parameter groups and semigroups

\[ T(\beta, a) = e^{-\beta H + i a \cdot P} \]

\[ U(R(\hat{n}, \psi)) = e^{i J \cdot \hat{n} \psi} \]

\[ W(\hat{n}, \psi) = e^{K \cdot \hat{n} \psi} \]
Domains for local symmetric semigroups
\{ H, P, J, K \}

Self-adjoint (on physical Hilbert space)

\[ H \geq 0 \] (Follows from reflection positivity)

Satisfy Poincaré commutation relations

No analytic continuation used!
Given a reflection positive Euclidean Green function or generating function we have:

- Hilbert space scalar product.
- A dense set of normalizable vectors.
- A representation of the Poincaré Lie algebra in terms of self-adjoint operators.
Comments:

• While analytic continuation is not used, reflection positivity ensures the existence of an analytic continuation.

• We can exploit the ability to calculate matrix elements of all operators in a dense set of normalizable states.

• $e^{-\beta H}$ and $H$ have the same eigenstates.
Particles: mass eigenstates

Dense set + Gram-Schmidt

\[ \Downarrow \]

Orthonormal basis of “wave functionals”

\[ B_n[\phi] \quad \langle B_n | B_m \rangle = \delta_{mn} \]

Solve for eigenstates in point spectrum of \( M^2 \)

\[ (M^2 B_\lambda)[\phi] = \lambda^2 B_\lambda[\phi] \]

\[ B_\lambda[\phi] = \sum_n b_n B_n[\phi] \]

\[ \sum_n \langle B_m | M^2 | B_n \rangle b_n = \lambda^2 b_m \]
Particles: mass-momentum eigenstates
(use translations and Fourier transforms)

\[ B_\lambda[\phi] = \sum_n b_n e^{i\phi(f_n)} \quad \text{(mass eigenfunctional)} \]

\[ B_\lambda(p)[\phi] = \int \frac{d^3 a}{(2\pi)^{3/2}} e^{-ip\cdot a} [T(0, a), B_\lambda][\phi] \]

\[ \langle C | B_\lambda(p) \rangle := \int \frac{d^3 a}{(2\pi)^{3/2}} e^{-ip\cdot a} \sum_{j=1}^{N_c} \sum_n b_n c_j^* Z[f_{n,a} - \theta g_j] \]
Particles: Mass-momentum-spin eigenstates (project on $SU(2)$ irreducible representations)

Normalize $B_\lambda(p)$

$$B_\lambda(p)[\phi] \quad \langle B_\lambda(p')|B_\lambda(p)\rangle = \delta(p' - p)$$

$$B_{\lambda,j}(p, \mu)[\phi] := \int_{SU(2)} dR[U(R), B_\lambda(R^{-1}p)][\phi] D_{\mu j}^{*}(R)$$

$$\langle C|B_{\lambda,j}(p, \mu)\rangle :=$$

$$\int \frac{d^3a dR}{(2\pi)^{3/2}} e^{-iR^{-1}p \cdot a} \sum_{j=1}^{N_c} \sum_{n} c_j^* b_n Z[f_{n,a,R} - \theta g_j] D_{\mu j}^{*}(R)$$

$$f_{n,a,R}(\tau, x) = f_n(\tau, Rx - a)$$
Finite Poincaré transformations of one-particle states ($\lambda \in \sigma_{pp}$)

One-particle subspaces are irreducible subspaces with respect to the Poincaré group

\[
\langle C| U[\Lambda, a]| B_{\lambda, j}(p, \mu) \rangle =
\]

\[
\sum_{\mu' = -j}^{j} \int dp' \langle C| B_{\lambda, j}(p', \mu') \rangle \mathcal{D}^{\lambda, j}_{\mu', p', \mu} [\Lambda, a]
\]

\[
\mathcal{D}^{\lambda, j}_{\mu', p', \mu} [\Lambda, a] =
\]

\[
\delta(\Lambda p - p') \sqrt{\frac{\omega_\lambda(p')}{\omega_\lambda(p)}} e^{-i\omega_\lambda(p') a^0 - ip' \cdot a} D^j_{\mu', \mu} [\Lambda_c^{-1}(\frac{p'}{\lambda}) \Lambda \Lambda_c(\frac{p}{\lambda})]
\]

\[
\omega_\lambda(p) = \sqrt{\lambda^2 + p \cdot p} \quad (p')^j = \Lambda^j_0 \omega_\lambda(p) + \Lambda^j_k p^k
\]

Here $\Lambda_c(\frac{p}{\lambda})$ is a rotationless Lorentz boost.
Scattering

- Time-dependent scattering has been used successfully to treat few-body problems (Kröger, Phys. Reports, 210(1992)46) in non-relativistic quantum mechanics.

- Calculations use wave packets and normalizable states.

- Haag-Ruelle scattering is a natural field theoretic generalization of non-relativistic time-dependent scattering theory (unlike LSZ it uses strong limits).

Construct

\[ J : \otimes \mathcal{H}_{\lambda_i,j_i} = \mathcal{H}_f \rightarrow \mathcal{H} \quad U_f[\Lambda, a] = \otimes U_{\lambda_i,j_i}[\Lambda, a] \]

The strong limit exists

\[ |\Psi_{\pm}\rangle = \lim_{t \to \pm \infty} e^{iHt} J e^{-iH_f t} |\psi_{f \pm}\rangle = \Omega_{\pm} |\psi_{f \pm}\rangle \]

The wave operators satisfy (Ruelle H.P.A. 35(1962)34)

\[ U[\Lambda, a] \Omega_{\pm} = \Omega_{\pm} U_f[\Lambda, a] \]
Structure of $J$

\[ B_{\lambda,j}(p, \mu)|0\rangle = |(\lambda,j)p, \mu\rangle \]

Creates one-particle state out of the vacuum

\[
\int J_i(p_i, \mu_i)f(p, \mu)d\mathbf{p} = \\
\int (-i\omega_{\lambda}(p)B_{\lambda,j}(p, \mu) - i[H, B_{\lambda,j}(p, \mu)])f(p, \mu)d\mathbf{p}
\]

(\cdots) selects “creation part” of $B_{\lambda,j}(p, \mu)$

\[ J(p_1, \mu_1, \cdots, p_n, \mu_n) = \prod J_i(p_i, \mu_i)|0\rangle \]
Wave functional representation

\[ B_{\lambda,j}(p, \mu) \rightarrow B_{\lambda,j}(p, \mu)[\phi] \]

\[ \int J_i(p_i, \mu_i)[\phi] f(p, \mu) dp = \]

\[ \int (-i \omega_\lambda(p) B_{\lambda,j}(p, \mu)[\phi] - i[H, B_{\lambda,j}(p, \mu)][\phi]) f(p, \mu) dp \]

\[ J[\phi] = \prod J_i(p_i, \mu_i)[\phi] \]
Two-space Scattering

\[ \lim_{t \to \pm \infty} \| (e^{-iHt} | \Psi_\pm \rangle - J e^{-iH_f t} | \Psi_{f \pm} \rangle) \| = 0 \]

\[ | \Psi_\pm \rangle = \lim_{t \to \pm \infty} e^{iHt} J e^{-iH_f t} | \Psi_{f \pm} \rangle = \Omega_{\pm} (H, J, H_f) | \Psi_f \rangle \]

\[ S_{fi} = \langle \Psi_+ | \Psi_- \rangle = \langle \Psi_{f+} | \Omega_+^\dagger (H, J, H_f) \Omega_- (H, J, H_f) | \Psi_{f-} \rangle \]

\[ = \lim_{t \to \infty} \langle \Psi_{f+} | e^{iH_f t} J^\dagger e^{-2iHt} J e^{iH_f t} | \Psi_{f-} \rangle \]
Kato-Birman invariance principle

\[ \lim_{t \to \pm \infty} \| (e^{-iHt} | \Psi_{\pm} \rangle - Je^{-iHf t} | \Psi_{f \pm} \rangle) \| = 0 \]

\[ \Downarrow \]

\[ \lim_{n \to \pm \infty} \| (e^{ine^{-\beta H}} | \Psi_{\pm} \rangle - Je^{ine^{-\beta H f}} | \Psi_{f \pm} \rangle) \| = 0 \]

Provides a possible computational strategy

\[ \langle \Psi_{f +} | S | \Psi_{f -} \rangle \approx \langle \Psi_{f +} | e^{-ine^{-\beta H f}} J^\dagger e^{2ine^{-\beta H}} Je^{-ine^{-\beta H f}} | \Psi_{f -} \rangle \]

\[ e^{2inx} \approx \sum_{m=0}^{N(n)} c_m(n)x^m \quad \rightarrow \quad e^{2ine^{-\beta H}} \approx \sum_{m=0}^{N(n)} c_m(n)e^{-m\beta H} \]

Convergence is \textbf{uniform} for each fixed \( n \)!
Matrix elements

\[
\langle B|e^{-m\beta H}|C\rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,m\beta,0} - \theta f_j]
\]

\[
C[\phi] = Je^{-ine^{-\beta H_f}} |\Psi_f\rangle[\phi]
\]

\[
B[\phi] = Je^{ine^{-\beta H_f}} |\Psi_f\rangle[\phi]
\]

Computable by quadrature in terms of \( Z[f] \) or \( \{S_n\} \)
Scattering in Euclidean space

- The results are standard Minkowski space results expressed in a representation where the Minkowski scalar product is evaluated in terms of a Euclidean generating functional.

- The time limits in the scattering theory are strong limits (compared with the weak limits used in LSZ scattering).

- The Haag-Ruelle scattering theory does not distinguish elementary and composite asymptotic states.

- Explicit representations of the Poincaré group exist for the bound and scattering states.

\[ U[\Lambda, a] \Omega_{\pm} = \Omega_{\pm} U_f[\Lambda, a] \]
Maiani-Testa No-Go Theorem (P.L.B. 245(1990)585)

\[ \langle 0 | \phi_\pi(p_1) \phi_\pi(p_2) J(0) | 0 \rangle \]

- Use LSZ interpolating fields for pions. Field creates more than 1-pion states from vacuum.
- Uses $\beta_1 \gg \beta_2 \gg 0$
- Uses Euclidean correlation functions.

This approach

- Uses Haag-Ruelle fields. Fields create only 1 pion states from vacuum - products approach exact scattering states in strong limit.
- $\beta$ is a fixed adjustable parameter ($H \leftrightarrow e^{-\beta H}$).
- Uses Minkowski scalar product.
- Calculations require wave packets, one-body solutions; no singularities, no analytic continuation.
Summary of formal results
Given a reflection positive generating functional

\[ \Downarrow \]

- Hilbert-space scalar product, \( \{H, P, J, K\} \)

- Single-particle states

- Scattering states, \( S \)-matrix elements.

- Finite Poincaré transformations on single-particle states and scattering states.
• Constructing a reflection positive Euclidean invariant generating functional is almost equivalent to constructing a non-trivial field theory (this must be relaxed for model applications).

• Practical calculations use a weakened form of reflection positivity (limited permutation symmetry).

• Full permutation symmetry = locality.

• Osterwalder-Schrader reconstruction of relativistic quantum mechanics does not require locality.
Green function approach - limited reflection positivity

\[ Z[f] = \sum_n \frac{i^n}{n!} S_n(f, \cdots, f) \]

\( x := (\tau, x) \quad \theta x = (-\tau, x) \quad f(x_1, \cdots, x_n) \in S_+ \)

\[ \int d^4x_1 \cdots d^4x_4 f_2^*(\theta x_2, \theta x_1) S_4(x_1, x_2; x_3, x_4)f_2(x_3, x_4) \geq 0 \]

\[ \int d^4x_1 d^4x_2 f_1^*(\theta x_1) S_2(x_1; x_2)f_1(x_2) \geq 0. \]

\[ S_4(x_1, x_2; x_3, x_4) = S_4(x_2, x_1; x_3, x_4) = S_4(x_1, x_2; x_4, x_3) \]
Green function representation

\[ Z \rightarrow S = \begin{pmatrix}
S_2(x; y) & S_3(x; y_1, y_2) & \ldots \\
S_3(x_1, x_2; y) & S_4(x_1, x_2; y_1, y_2) & \vdots \\
\vdots & \vdots & \ddots \\
\end{pmatrix} \]

\[ B[\phi] \rightarrow \begin{pmatrix}
f_1(x_{11}) \\
f_2(x_{21}, x_{22}) \\
\vdots \\
\end{pmatrix} \]

\[ \langle B | C \rangle = (\theta f_B, Sf_C)_e \]
\[ \langle x|H|f \rangle := \{0, \frac{\partial}{\partial x_{11}^0} f_1(x_{11}), \left( \frac{\partial}{\partial x_{21}^0} + \frac{\partial}{\partial x_{22}^0} \right) f_2(x_{21}, x_{22}), \cdots \} \]

\[ \langle x|P|f \rangle := \{0, -i \frac{\partial}{\partial x_{11}} f_1(x_{11}), -i \left( \frac{\partial}{\partial \mathbf{x}_{21}} + \frac{\partial}{\partial \mathbf{x}_{22}} \right) f_2(x_{21}, x_{22}), \cdots \} \]

\[ \langle x|J|f \rangle := \{0, -i \mathbf{x}_{11} \times \frac{\partial}{\partial \mathbf{x}_{11}} f_1(x_{11}), -i \left( \mathbf{x}_{21} \times \frac{\partial}{\partial \mathbf{x}_{21}} + \mathbf{x}_{22} \times \frac{\partial}{\partial \mathbf{x}_{22}} \right) f_2(x_{21}, x_{22}), \cdots \} \]

\[ \langle x|K|f \rangle := \{0, \left( \mathbf{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11}^0 \frac{\partial}{\partial \mathbf{x}_{11}} \right) f_1(x_{11}), \left( \mathbf{x}_{21} \frac{\partial}{\partial x_{21}^0} - x_{21}^0 \frac{\partial}{\partial \mathbf{x}_{21}} + \mathbf{x}_{22} \frac{\partial}{\partial x_{22}^0} - x_{22}^0 \frac{\partial}{\partial \mathbf{x}_{22}} \right) f_2(x_{21}, x_{22}), \cdots \}. \]
Modifications for spin

\[ J : \left(-i\vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}}\right) \rightarrow \left(-i\vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} + \vec{\Sigma}\right) \]

\[ K : \left(\vec{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11}^0 \frac{\partial}{\partial \vec{x}_{11}}\right) \rightarrow \left(\vec{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11}^0 \frac{\partial}{\partial \vec{x}_{11}} + \vec{B}\right) \]

where

\[ \vec{\Sigma} = i\vec{\nabla}_\phi D(e^{-\frac{i}{2} \vec{\sigma} \cdot \vec{\phi}}, e^{\frac{i}{2} \vec{\sigma}^t \cdot \vec{\phi}})_{aa'} \]

and

\[ \vec{B} = \vec{\nabla}_\rho D(e^{-\frac{i}{2} \vec{\sigma} \cdot \vec{\rho}}, e^{\frac{i}{2} \vec{\sigma}^t \cdot \vec{\rho}})_{aa'} \]

and \( D(g_1, g_2) \) is a representation of \( SU(2) \times SU(2) \).
Two-body scattering

\[ S_2(x; y) \quad K(x_1, x_2; y_2, y_1) \]

\[ S_0 = S_2(x_1; y_1)S_2(x_2; y_2) \]

\[ S_4 = S_0 + S_0KS_4 \]

\[ S = \begin{pmatrix} S_2(x; y) & 0 \\ 0 & S_4(x_1, x_2; y_1, y_2) \end{pmatrix} \]

\[ \langle C|B\rangle = (\theta g_1, S_2 f_1)_e + (\theta g_2, S_4 f_2)_e \]
Particles - scalar
(case of free $S_2$)

$$(f, \theta S_2 f)_e$$

$$= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{ip \cdot (\theta x - y)}}{p^2 + m^2} f(y)$$

$$= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{-ip_0 \cdot (x_0 + y_0) + i\vec{p} \cdot (x - y)}}{(p^0 + i\omega_m(\vec{p}))(p^0 - i\omega_m(\vec{p}))} f(y)$$

$$= \int d^3p \frac{|g(\vec{p})|^2}{2\omega_m(\vec{p})} \geq 0$$

where

$$g(\vec{p}) := \frac{1}{(2\pi)^{3/2}} \int d^4y f(y) e^{-\omega_m(\vec{p}) y_0 - i\vec{p} \cdot \vec{y}}.$$
Particles - fermions
(Euclidean time reversal has a spinor component)

\[(f, \theta \gamma^0 S_2 f)_e\]

\[= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) e^{ip \cdot (\theta x - y)} \gamma^0 \frac{m - p \cdot \gamma^e}{p^2 + m^2} f(y)\]

\[= \int g^\dagger(\vec{p}) \frac{\Lambda_+(p)}{(2\pi)^3} g(\vec{p}) d^3p\]

where

\[\Lambda_+(p) := \frac{\omega_m(\vec{p}) + \gamma^0 \vec{\gamma} \cdot \vec{p} - m\gamma^0}{2\omega_m(\vec{p})}\]
Scattering in Euclidean space

**Approximation 1:** Use sharply peaked (in momentum) normalizable states to approximate plane-wave on-shell transition matrix elements.

\[
\langle \Psi_{f+} | S | \Psi_{f-} \rangle = \langle \Psi_{f+} | \Psi_{f-} \rangle - 2\pi i \langle \Psi_{f+} | \delta(E_+ - E_-) \, T | \Psi_{f-} \rangle
\]

\[
\langle p_1', \mu_1', p_2', \mu_2' | T | p_1, \mu_1, p_2, \mu_2 \rangle \approx \frac{\langle \Psi_{f+} | S | \Psi_{f-} \rangle - \delta_{ab} \langle \Psi_{f+} | \Psi_{f-} \rangle}{2\pi i \langle \Psi_{f+} | \delta(E_+ - E_-) | \Psi_{f-} \rangle}
\]
Scattering injection operators (N=2)

**Approximation 2:** Calculate $\psi_\lambda(x, \tau; p, \mu)$

- $\psi_\lambda(x, \tau; p, \mu)$: eigenstate of $M^2, P, j^2, j_z$ in one-body Hilbert space generated by $S_2$.

$$J_i(x, \tau; p, \mu) := (-i \omega_\lambda(p) + i \frac{\partial}{\partial \tau}) \psi_\lambda(x, \tau; p, \mu)$$

$$J(x_1, \tau_1, x_2, \tau_2; p_1, \mu_1, p_2, \mu_2) = J_1(x_1, \tau_1; p_1, \mu_1) J_2(x_2, \tau_2; p_2, \mu_2)$$
Scattering in Euclidean space

Use time-dependent scattering to calculate $S$ matrix elements in normalizable states.

Use Kato-Birman invariance principle to express $S$ in terms of $e^{-\beta H}$.

\[
\langle \Psi_{f+} | S | \Psi_{f-} \rangle
\]

\[
= \lim_{t \to \infty} \langle \Psi_{f+} | e^{iH_f t} J^\dagger e^{-2iHt} Je^{iH_f t} | \Psi_{f-} \rangle
\]

\[
= \lim_{n \to \infty} \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger e^{2i\beta H} Je^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle
\]
Scattering in Euclidean space

**Approximation 3:** Replace $n$ by large fixed $n$.

$$
\langle \Psi_{f+} | S | \Psi_{f-} \rangle \\
\approx \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle
$$
Approximation 4: Uniform polynomial approximation

\[ e^{2i\beta H} \approx \sum c_m(n)(e^{-\beta mH}) \]

**note**  \( \sigma(e^{-\beta H}) \in [0, 1] \) (compact)

\[ e^{2inx} \approx \sum c_m(n)x^m \quad x \to e^{-\beta H} \]

\[ |e^{2inx} - \sum c_m(n)x^m| < \epsilon(n) \quad \forall x \in [0, 1] \]

\[ \|\left[e^{2i\beta H} - \sum c_m(n)(e^{-\beta mH})\right]\psi\| < \epsilon(n)\|\psi\| \]
\[ f(x) \approx \frac{1}{2} c_0 T_0(x) + \sum_{k=1}^{N} c_k T_k(x) \]

\[ c_j = \frac{2}{N + 1} \sum_{k=1}^{N} f(\cos\left(\frac{2k - 1}{N + 1} \pi/2\right)) \cos(j \frac{2k - 1}{N + 1} \pi/2) \]

\[ f(e^{-\beta H}) \approx \frac{1}{2} c_0 T_0(e^{-\beta H}) + \sum_{k=1}^{N} c_k T_k(e^{-\beta H}) \]

\[ f(x) = e^{2inx} \]

\[ |e^{2inx} - P_N(x)| < 2 \frac{n^{N+1}}{(N + 1)!} \]
\[ \langle \Psi_{f+} | S | \Psi_{f-} \rangle \approx \]

\[ = \sum c_m(n) \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^{\dagger} (e^{-\beta mH}) J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle \]

Each approximation converges - the order of the approximations is important (1) → (2) → (3) → (4).
Three and four-body phenomenology and cluster properties

\[ S^{-1}(123) = \]
\[ S^{-1}(1)S^{-1}(2)S^{-1}(3) - K(12)S^{-1}(3) - \]
\[ K(23)S^{-1}(1) - K(31)S^{-1}(2) - K(123) \]

\[ S^{-1}(1234) = \]
\[ S^{-1}(1)S^{-1}(2)S^{-1}(3)S^{-1}(4) + K(12)K(34) + K(13)K(24) + \]
\[ K(14)K(23) - K(12)S^{-1}(3)S^{-1}(4) - K(13)S^{-1}(2)S^{-1}(4) + \]
\[ -K(14)S^{-1}(2)S^{-1}(3) - K(23)S^{-1}(1)S^{-1}(4) + \]
\[ -K(24)S^{-1}(1)S^{-1}(3) - K(34)S^{-1}(1)S^{-1}(2) + \cdots \]
Test of method: non-relativistic separable potential
(solvable so all approximations can be tested)

\[ H = \frac{k^2}{m} - |g\rangle \lambda \langle g| \]

\[ \langle k | g \rangle = \frac{1}{m^2_{\pi} + k^2} \]

calculate \( \langle k' | T(k^+) | k \rangle \) using matrix elements of \( e^{-\beta H} \) in normalizable states.
Approximation 3:

Imaginary S vs n-limit - 1 GeV
Approximation 3:

Real $S-1$ vs n-limit - 1 GeV
Approximation 3:

Im S vs n-limit - 2.1 GeV
Approximation 4:

Degree 300 polynomial compared to $e^{-inx}$, $n = 220$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\Delta \cos(nx)$</th>
<th>$\Delta \sin(nx)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$4.44089 \times 10^{-16}$</td>
<td>$8.32667 \times 10^{-15}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$2.35367 \times 10^{-14}$</td>
<td>$1.46966 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$5.55112 \times 10^{-16}$</td>
<td>$3.6797 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$3.84137 \times 10^{-14}$</td>
<td>$1.80689 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$1.72085 \times 10^{-14}$</td>
<td>$1.32672 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$2.77556 \times 10^{-15}$</td>
<td>$2.93793 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$6.66134 \times 10^{-16}$</td>
<td>$3.33344 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$8.54872 \times 10^{-15}$</td>
<td>$2.50355 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$1.02141 \times 10^{-14}$</td>
<td>$1.35447 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$1.22125 \times 10^{-15}$</td>
<td>$2.72282 \times 10^{-14}$</td>
</tr>
<tr>
<td>1</td>
<td>$4.88498 \times 10^{-15}$</td>
<td>$6.61415 \times 10^{-14}$</td>
</tr>
</tbody>
</table>
Real part of $<k|T(k)|k>$  (exact - black, polynomial - red)
Im part of $<k|T(k)|k>$ (exact - black, polynomial - red)
Conclusions - Outlook

- Phenomenology based on model reflection-positive Euclidean Green functions can be used to formulate a relativistic quantum theory.
- Analytic continuation is not necessary.
- The Poincaré invariant S-matrix. Cluster properties are easily satisfied for fixed $N$.
- Models can be motivated by field-theory based phenomenology.
- A test using an exactly solvable model suggests that GeV scale scattering calculations are possible in this framework.
Future directions

- Euclidean BS free $S_2$.

- Euclidean BS $S_2$ with continuous Lehmann weight.

- Nakanishi representation and reflection positivity.

- Current matrix elements.
Thanks!

- Workshop organizers
- Nuclear Theory Center
- U.S. D.O.E. Office of Science