Applications of AdS/CFT correspondence to cold atom physics

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Outline

• Basics of AdS/CFT correspondence
• Schrödinger group and correlation functions
• Nonrelativistic AdS/CFT and cold atoms
• Holographic Efimov effect
Holographic principle

• We have some evidence that

Quantum field theory in \(d\) spacetime dimensions \(\leftrightarrow\) Quantum gravity in higher dimensions

• Holographic radial direction – RG scale
AdS spacetime and its relation to CFT

- Einstein-Hilbert gravity action

\[ S = \frac{1}{2\kappa} \int dz d^d x \sqrt{-g} \left( R + \frac{d(d - 1)}{L^2} \right) \]

- Most symmetric solution is Anti-de Sitter spacetime

\[ ds^2 = L^2 \frac{dz^2 + \eta_{\mu \nu} x^\mu x^\nu}{z^2} \]

- Isometry group of $AdS_{d+1}$ is $SO(d, 2)$
- Conformal group in $Mink_d$ is also $SO(d, 2)$!
AdS/CFT correspondence

\[ \mathcal{N} = 4, \ SU(N_c) \]

in \( d = 4 \) Minkowski \( \leftrightarrow \)

IIB string theory

in \( AdS_5 \times S_5 \)

- Parameters on gauge side: \( g_{YM} \) and \( N_c \)
- Parameters on gravity side: \( L, l_{st} \) and \( \kappa \)

- Mapping

\[
\frac{L}{l_{st}} \sim g_{YM}^2 N_c \quad \frac{L^8}{\kappa} \sim N_c^2
\]

- It is a weak/strong duality
- Classical supergravity is valid if \( N_c^2, g_{YM}^2 N_c \gg 1 \)
### Holographic dictionary

<table>
<thead>
<tr>
<th>Boundary QFT</th>
<th>Bulk gravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>operator $\mathcal{O}$</td>
<td>dynamical field $\phi$</td>
</tr>
<tr>
<td>scaling dimension $\Delta_\mathcal{O}$</td>
<td>mass $m_\phi$</td>
</tr>
<tr>
<td>global symmetry</td>
<td>gauge symmetry</td>
</tr>
<tr>
<td>finite $T$</td>
<td>Hawking $T$ of a black hole</td>
</tr>
<tr>
<td>entropy</td>
<td>Hawking entropy</td>
</tr>
<tr>
<td>chemical potential</td>
<td>U(1) gauge field</td>
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</tbody>
</table>

- Correlation functions can be calculated from

$$\langle e^{\int \mathcal{O} \phi} \rangle_{\text{CFT}} = Z_{\text{gravity}}[\phi \to \phi_0]$$
Schrödinger group

- Spacetime symmetries of free Schrödinger/diffusion equation form Schrödinger group $Sch(D)$  Niederer 72, Hagen 72
- Schrödinger group comprises
  - translations $P_i$ and $H$
  - spatial rotations $M_{ij}$
  - Galilean boosts $K_i$
  - number operator $N$
  - scale transformation $D \Rightarrow$ dynamical exponent $z = 2$
  - special conformal transformation $C$
- Theory symmetric under $Sch(D)$ is called NRCFT
- Example: two particles with conformal $1/r^2$ potential
Schrödinger group

- Nonrelativistic primary operators $\mathcal{O}$ have well-defined scaling dimension $\Delta_\mathcal{O}$ and particle number $N_\mathcal{O}$ \cite{Nishida&Son08}

\[ [D, \mathcal{O}] = i\Delta_\mathcal{O} \quad [N, \mathcal{O}] = N_\mathcal{O} \]

and commute with Galilean boosts $K_i$ and special conformal transformation $C$

\[ [K_i, \mathcal{O}] = 0 \quad [C, \mathcal{O}] = 0 \]

Descendants are formed by commutators with $P_i$ and $H$

- Kinematic invariants of $Sch(D)$ group \cite{Volovich&Wen09}
  - mixed invariants $\nu_{ijn} = \frac{(\vec{x}_{intjn} - \vec{x}_{jn}t_{in})^2}{2t_{ij}t_{intjn}}$ \quad $i < j < n$
  - time cross-ratios $\frac{t_{ij}t_{kl}}{t_{ik}t_{jl}}$
Correlators in Euclidean NRCFT

- Schrödinger symmetry and causality impose constraints on the correlators of the primary fields
  
- 2-point function is fixed up to a constant \((\bar{x} = (\vec{x}, t))\)

\[
G_2(\bar{x}_1, \bar{x}_2) = C \delta_{\Delta_1, \Delta_2} \delta_{M_1, M_2} \theta(t_{12}) t_{12}^{-\Delta_1} \exp \left[ -\frac{M_1}{2} \frac{\bar{x}_{12}^2}{t_{12}} \right]
\]

- 3-point function is determined up to a function \(\Psi(v_{123})\)

\[
G_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \delta_{M_1+M_2, M_3} \theta(t_{13}) \theta(t_{23}) \prod_{i<j} t_{ij}^{-\Delta_{ij,n}/2} \times \\
\exp \left[ -\frac{M_1}{2} \frac{\bar{x}_{13}^2}{t_{13}} - \frac{M_2}{2} \frac{\bar{x}_{23}^2}{t_{23}} \right] \Psi(v_{123})
\]

- 4-point function is determined up to a non-universal function

\(\Psi\left(\frac{t_{12}t_{34}}{t_{14}t_{32}}, v_{124}, v_{134}, v_{234}\right)\)
Two-component fermions at unitarity

• Vacuum theory is defined by action

\[ S[\psi, \phi] = \int dt d^Dx \left[ \sum_{i=1}^{2} \psi_i^*(i\partial_t + \frac{\Delta}{2m})\psi_i - \frac{1}{c_0} \phi^* \phi + (\phi^* \psi_1 \psi_2 + \phi \psi_2^* \psi_1^*) \right] \]

• Bare parameter $c_0$ is related to scattering length $a$ and cut-off $\Lambda$

• Unitarity regime $a^{-1} \to 0 \Rightarrow$ no intrinsic length scale in theory

• Exact propagators and scaling dimensions of $\psi$ and $\phi$ can be determined analytically

• Fermions at unitarity are believed to be symmetric under the full Schrödinger group 

Mehen et al. 2000

• It is believed to be NRCFT (primary operators, operator/state correspondence, conformal Ward identities) 

Nishida&Son 08
Nonrelativistic AdS/CFT

• NRCFT in $D$ dimensions $\iff$ gravity theory in higher dimensions?

• Schrödinger algebra $Sch(D)$ is a subalgebra of the conformal algebra $so(D + 2, 2)$ in $Mink_{D+2}$ that commutes with the light-cone momentum $P^+$

• Deformation of $AdS_{D+3}$ in light-cone coordinates leads to the new $Sch_{D+3}$ metric

$$ds^2 = -\frac{dt^2}{z^4} + \frac{-2dtd\xi + dx^i dx^i + dz^2}{z^2} \quad i = 1, \ldots, D$$

Son 08, Balasubramanian&McGreevy 08

• Isometries of $Sch_{D+3}$ obey the Schrödinger algebra $Sch(D)$

• $\partial_\xi$ corresponds to the particle number generator $N$ and coordinate $\xi$ is possibly compact
3-point function $G_3$ in cold atoms

- We compute $G_3 = \langle \psi_1(\vec{x}_1)\psi_2(\vec{x}_2)\phi^*(\vec{x}_3) \rangle$ at unitarity regime in position space by performing integration over $\vec{x}$

- Two important points
  1. no condensate in non-relativistic vacuum $\langle \psi \rangle = 0$, $\langle \phi \rangle = 0$
  2. Yukawa vertex is not renormalized in vacuum

- $G_3$ is completely determined by $\Delta_\psi = \frac{D}{2}$ and $\Delta_\phi = 2$
3-point function $G_3$

- Agreement with Schrödinger Ward identities
  
  - We determined the non-universal scaling function for $D > 2$
    
    $$
    \Psi(y) \sim y^{\frac{D}{2}+1} \gamma\left(\frac{D}{2} - 1, y\right),
    $$

    where $\gamma(n, y) = \int_0^y t^{n-1} e^{-t} dt$

- AdS/CFT gives non-universal scaling function

  $$
  \Psi(y) \sim \int_{\mathbb{R}+i\epsilon} dv \int_{\mathbb{R}+i\epsilon'} dv' e^{-iM_1 v - iM_2 v'} \times
  $$
  
  $$(v - v' + i y)^{-\Delta_{12,3}/2} (v')^{-\Delta_{23,1}/2} v^{-\Delta_{13,2}/2},$$

  where $\Delta_{i,j,k} = \Delta_i + \Delta_j - \Delta_k$
3-point function $G_3$ from AdS/CFT

- We take unitarity scaling $\Delta_1 = \Delta_2 = D/2$, $\Delta_3 = 2$ and perform double contour integration

$$\Psi(y) \sim y^{-\frac{D}{2} + 1} \gamma\left(\frac{D}{2} - 1, y\right)$$

- Non-universal scaling function $\Psi(y)$ agrees with unitarity cold atoms result! [Fuertes & SM 09]

- We take free scaling dimension $\Delta_1 = \Delta_2 = D/2$ and $\Delta_3 = D$ and perform double contour integration

$$\Psi(y) = \text{const}$$

- Agreement with free QFT

- Possibly AdS/CFT describes both free and unitarity regime [Son 08]
Bosons at unitarity

- The action is similar

$$S[\psi, \phi] = \int dt d^D x \left[ \psi^* \left( i \partial_t + \frac{\Delta}{2m} \right) \psi - \frac{1}{c_0} \phi^* \phi \right.$$  

$$+ (\phi^* \psi \psi + \phi \psi^* \psi^*) \right]$$

- Unitarity regime $a^{-1} \rightarrow 0 \Rightarrow$ no intrinsic length scale in theory

- Can be prepared in cold atoms experiments, e.g. $^7\text{Li}, ^{133}\text{Cs}$...

- Two-body problem is similar to fermions

- It is not NRCFT due to the Efimov effect
Three-body problem and the Efimov effect

Energy spectrum near the unitarity regime

- At unitarity $a = \pm \infty$ spectrum becomes geometric

\[ \frac{E_T^{(n+1)}}{E_T^{(n)}} \rightarrow e^{-2\pi/s_0} \quad \text{as } n \rightarrow \infty \quad s_0 \approx 1.0062 \]

- The spectrum is manifestation of scale quantum anomaly

- In RG language $\rightarrow$ limit cycle solution
Breitenlohner-Freedman bound in $AdS_{d+1}$

- Free complex scalar

$$S[\phi, \phi^*] = -\int dz d^d x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + m^2 \phi^* \phi \right)$$

in $AdS_{d+1}$ spacetime

$$ds^2 = \frac{dz^2 + \eta^{\mu\nu} dx_\mu dx_\nu}{z^2}$$

- Fourier transform $x^\mu \rightarrow q^\mu$ on the boundary and change variables $\psi = z^{(1-d)/2} \phi$

$$-\partial_z^2 \psi + \frac{m^2 + \frac{d^2-1}{4}}{z^2} \psi = -q^2 \psi, \quad q^2 \equiv -(q^0)^2 + \vec{q}^2$$

- Map onto 1D QM problem with inverse square potential!
Inverse square potential in QM

\[-\partial_z^2 \psi - \frac{\kappa}{z^2} \psi = E \psi\]

• The potential is singular and must be regularized

• Two branches of solution
  
  • \(\kappa < \kappa_{cr} = \frac{1}{4}\) → no bound states, continuous spectrum
  
  • \(\kappa > \kappa_{cr}\) → infinite geometric bound state spectrum

• In our mapping

\[E < 0 \Rightarrow (q^0)^2 < 0\]

\[\kappa > \kappa_{cr} \Rightarrow m^2 < m_{BF}^2 = -\frac{d^2}{4}\]

• The bound was first derived from positivity of conserved energy functional of scalar fluctuations  Breitenlohner&Freedman 82
No BF bound in $Sch_{D+3}$

- Free complex scalar in $Sch_{D+3}$

$$S[\phi, \phi^*] = - \int dz dt d\xi d^Dx \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + m_0^2 \phi^* \phi \right)$$

- Mapping onto Schrödinger equation

$$-\partial^2_z \psi + \frac{m^2 + \frac{(D+2)^2-1}{4}}{z^2} \psi = -\tilde{q}^2 \psi, \quad \tilde{q}^2 \equiv -2M\omega + \vec{q}^2$$

- Due to nonrelativistic dispersion

$$E < 0 \Rightarrow \omega < 0$$

- Nothing special happens at $m^2 = m_{BF}^2 = -\frac{(D+2)^2}{4}$

- No stability bound in nonrelativistic AdS/CFT!
Two-point correlator $\langle OO^\dagger \rangle$ for $m^2 < m_{BF}^2$

- Using standard AdS/CFT machinery we can calculate

$$\langle OO^\dagger \rangle \sim \tan \{ |\nu| \ln \tilde{q} + \gamma \},$$

where $\nu = \sqrt{\frac{(D+2)^2}{4} + m^2}$ and $\tilde{q}^2 \equiv -2M\omega + \bar{q}^2$

- Properties
  - $\langle OO^\dagger \rangle$ is log-periodic in $\tilde{q}$
  - Operator $O$ describes infinitely many particles

$$\frac{\omega_{n+1}}{\omega_n} = \exp \left( -\frac{2\pi}{|\nu|} \right)$$

- Continuous scale symmetry is broken $\rightarrow$ limit cycle solution
- $\gamma$ determines initial UV position on RG limit cycle
Limit cycles in QM and complex $\Delta$

- Efimov effect
  - Trimer operator $O = \psi \phi$ has
    \[
    \Delta_{\pm} = \frac{5}{2} \pm i s_0
    \]

- QM with $1/r^2$ potential in $D$ dimensions
  - For $\kappa > \kappa_{cr} = \frac{(D-2)^2}{4}$, composite $O = \psi \psi$ acquires complex scaling dimension
    \[
    \Delta_{\pm} = \frac{D + 2}{2} \pm \sqrt{\frac{(D - 2)^2}{4} - \kappa}
    \]

- If described by AdS/CFT $\rightarrow m^2 < m^2_{BF}$
Conclusion and outlook

- AdS/CFT was extended to nonrelativistic physics
- Schrödinger symmetry is powerful
- Agreement of specific 3-point function, but better understanding?
- Limit cycles can be realized in nonrelativistic AdS/CFT
- Calculate limit cycle two-point function in QM and compare with holographic prediction
Extra slides
Applications to condensed matter physics

- Holographic systems with Schrödinger symmetry
- Holographic superfluids
- Holographic non-Fermi liquids
- Holographic systems with Lifshitz symmetry
Schrödinger algebra

- Centrally extended Galilei algebra

\[
[M_{ij}, M_{kl}] = i(\delta_{ik} M_{jl} - \delta_{jk} M_{il} + \delta_{il} M_{kj} - \delta_{jl} M_{ki}) ,
\]

\[
[M_{ij}, K_k] = i(\delta_{ik} K_j - \delta_{jk} K_i) , \quad [M_{ij}, P_k] = i(\delta_{ik} P_j - \delta_{jk} P_i) ,
\]

\[
[P_i, K_j] = -i\delta_{ij} N , \quad [H, K_j] = -iP_j .
\]

- Additionally

\[
[P_i, D] = -iP_i \quad , \quad [P_i, C] = -iK_i \quad , \quad [K_i, D] = iK_i \quad ,
\]

\[
[D, C] = -2iC \quad , \quad [D, H] = 2iH \quad , \quad [C, H] = iD .
\]

- The generators $H$, $D$ and $C$ close a subalgebra $sl(2, R)$