Renormalising nuclear forces

or

How can we build an effective Hamiltonian for nuclear physics?

Mike Birse
The University of Manchester

Thanks to the INT, Seattle, and the organisers of the program INT-09-1 “Effective field theories and the many-body problem”, April–June 2009
What’s the point of an effective (field) theory?

- no model assumptions – just low-energy degrees of freedom and symmetries
- estimates of errors and theory will tell you if it breaks down (no convergence)
- consistency of effective operators and interactions
- effective coupling constants are “universal”
  → links between different low-energy phenomena
    (\(c_i\)’s: \(\pi N\) scattering \(\leftrightarrow\) two-pion exchange forces)
  → bridges between low-energy observables and underlying theory
    (scattering lengths: scattering processes \(\leftrightarrow\) lattice QCD)
How does it work?

• systematic expansion in powers of ratios of low-energy scales $Q$
  (momenta, $m_\pi$, $m_\rho$, $M_N$, $4\pi F_\pi$, ... $\sim 200$ MeV)
  to scales of underlying physics $\Lambda_0$
  ($m_\rho$, $M_N$, $4\pi F_\pi$, ... $\gtrsim 700$ MeV?)

• interactions with ranges $\sim 1/\Lambda_0$
  not resolved at scales $Q$.
  → replaced by contact interactions.

• iterations (loop diagrams) usually infinite
  → need to renormalise.

• works provided we have a consistent expansion
  (otherwise trying to renormalise an infinite number of constants,
   simultaneously)
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Where does it work?

Works well for purely pionic and $\pi N$ systems

- pions $\sim$ Goldstone bosons of hidden chiral symmetry
- strong interactions weak at low energies
  $\rightarrow$ chiral perturbation theory
- terms organised by naive dimensional analysis
  aka “Weinberg power counting”
  (simply counts powers of low-energy scales – momenta and $m_\pi$)
What’s the problem with building an EFT for nuclear forces?

Chiral perturbation theory

- simply counting powers of low-energy scales: perturbative
- works for weakly interacting systems (eg pions, photons and $\leq 1$ nucleon)
- but nucleons interact strongly at low-energies
- bound states exist (nuclei!)

→ need to treat some interactions nonperturbatively
Basic nonrelativistic loop diagram

\[
\frac{M}{(2\pi)^3} \int \frac{d^3q}{p^2 - q^2 + i\epsilon} = -i \frac{Mp}{4\pi} + \text{analytic}
\]

- of order $Q$ [Weinberg (1991)]
- but potential starts at order $Q^0$
  (OPE and simplest contact interaction)
- each iteration suppressed by power of $Q/\Lambda_0$
  $\rightarrow$ perturbative provided $Q < \Lambda_0$
- integral linearly divergent
  $\rightarrow$ cut off (or subtract) at $q = \Lambda$
- contributions multiplied by powers of $\Lambda/\Lambda_0$
  $\rightarrow$ again perturbative provided $\Lambda < \Lambda_0$
Workaround: “Weinberg prescription”

- expand potential to some order in $Q$
- then iterate to all orders in favourite dynamical equation (Schrödinger, Lippmann-Schwinger, . . .)
- widely applied and even more widely invoked

but no clear power counting for observables
resums subset of terms to all orders in $Q$ (and some of these depend on regulator)
not necessarily a problem if these terms are small
but what if we rely on them to generate bound states?
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Has led to vigorous debate over the last 12+ years

EFT community has polarised around two philosophies:

- **Orthodox**
  "The Prophet of EFT gave us the Power Counting in the holy texts, Phys Lett B251 and Nucl Phys B363."

- **Liberal**
  "Let the renormalisation group decide!"

and the orthodox party seems to be winning the election, so far...
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How can we iterate interactions consistently?

Identify new low-energy scales

- promote leading-order terms to order $Q^{-1}$
  (cancels $Q$ from loop $\rightarrow$ iterations not suppressed)
- can, and must, then be iterated to all orders
  (all other terms: perturbations)

Examples of new scales

- S-wave scattering lengths $1/a \lesssim 40$ MeV
  [van Kolck; Kaplan, Savage and Wise (1998)]
  $\rightarrow$ for $p < m_\pi$:
  "pionless EFT"
  $\equiv$ effective-range expansion
  [Schwinger (1947); Bethe (1949)]
- also atomic systems with Feshbach resonance close to threshold
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One-pion exchange

- important for nuclear physics at energies $\sim 100$ MeV
- order $Q^0$ in chiral counting
  $\rightarrow$ treat as a perturbation [Kaplan, Savage and Wise (1998)]
- $S$ waves: series coverges slowly, if at all
- OPE “unnaturally” strong
  (cf success of older phenomenology and Weinberg prescription)
- strength of OPE set by scale
  $$\lambda_{NN} = \frac{16\pi F^2_\pi}{g_A^2 M_N} \simeq 290 \text{ MeV}$$
  built out of high-energy scales $(4\pi F_\pi, M_N)$ but $\sim 2m_\pi$
  $\rightarrow$ another low-energy scale?
How do we analyse scale-dependence of strongly-interacting systems?

General tool for this: the renormalisation group

- scattering by contact interactions is ill-defined in QM
- couple to virtual states with arbitrarily high momenta
- example: basic loop diagram for $S$ waves behaves as

$$\frac{M}{(2\pi)^3} \int \frac{d^3q}{p^2 - q^2 + i\varepsilon} \sim -\frac{M}{2\pi^2} \int dq \quad \text{for large } q$$

(linear divergence)

→ need to renormalise
Procedure

- **identify all relevant low-energy scales** $Q$
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• rescale: express all dimensioned quantities in units of $\Lambda$ (potential and all low-energy scales)
Follow flow of effective potential as $\Lambda \to 0$

→ look for fixed points
  • rescaled theories independent of $\Lambda$
  • correspond to scale-free systems
  • endpoints of RG flow

- stable fixed point
- unstable fixed point
Expand around fixed point using perturbations that scale like $\Lambda^\nu$

- $\nu < 0$ relevant or superrenormalisable
  (unstable; eg masses in QFTs)
- $\nu > 0$ irrelevant or nonrenormalisable
  (stable; eg mesonic ChPT)
- $\nu = 0$ marginal or renormalisable
  (→ $\ln \Lambda$ scale dependence; eg couplings in QED, QCD)

→ EFT with power counting: $Q^d$ where $d = \nu - 1$
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$\rightarrow$ EFT with power counting: $Q^d$ where $d = \nu - 1$

\(\Lambda\) is highest acceptable low-energy scale

- order $Q$
- rescaling $\rightarrow$ power of $\Lambda$ counts low-energy scales
What does the RG tell us about short-range potentials?

RG equation for $\hat{V}(\hat{k}', \hat{k}, \hat{p}; \Lambda)$ (rescaled)

$$\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = \hat{p} \frac{\partial \hat{V}}{\partial \hat{p}} + \hat{k}' \frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k} \frac{\partial \hat{V}}{\partial \hat{k}} + \hat{V} + \hat{V}(\hat{k}', 1, \hat{p}; \Lambda) \frac{1}{1 - \hat{p}^2} \hat{V}(1, \hat{k}, \hat{p}; \Lambda)$$

$p, k, k'$: on- and off-shell momenta (low-energy scales $Q$)
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Two fixed points

- **trivial** $V_0 = 0 \rightarrow$ free particles
- **nontrivial** [Birse, McGovern, Richardson (1998)]
  $\rightarrow$ “unitary limit” (bound state at threshold, $a \rightarrow \infty$)
- both scale-free systems
Trivial fixed point

Expansion around \( V_0 = 0 \) in powers of momenta

\[
V(p) = C_0 + C_2 p^2 + C_4 p^4 + \cdots
\]

- \( p^{2n} \) are RG eigenfunctions
- orders given by naive (Weinberg) counting: \( Q^0, Q^2, Q^4, \ldots \)
- coefficients \( C_{2n} \) related to expansion of on-shell K matrix
  (like T matrix but standing-wave bc’s)
- perturbative
- appropriate EFT for thermal \( np \) scattering
  and other systems without low-energy bound/virtual states
Nontrivial fixed point

\[ V_0(p, \Lambda) = -\frac{2\pi^2}{M\Lambda} \left[ 1 - \frac{p}{2\Lambda} \ln \frac{\Lambda + p}{\Lambda - p} \right]^{-1} \]  

(sharp cutoff)

- order \( Q^{-1} \) (so must be iterated)
- exactly cancels basic loop integral in LS equation

\[ T(p) = i \frac{4\pi}{M \rho} \]  

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Expanding around this point

\[ V(p, \Lambda) = V_0(p, \Lambda) + V_0(p, \Lambda)^2 \frac{M}{4\pi} \left( -\frac{1}{a} + \frac{1}{2} r_e p^2 + \cdots \right) \]

- factor \( V_0^2 \propto \Lambda^{-2} \) promotes terms by two orders compared to naive expectation: \( Q^{-2}, Q^0, \ldots \)
- coefficients of perturbations directly related to observables: effective-range expansion
Enhancement follows from form of wave functions as $r \to 0$

**Two particles in unitary limit**

- irregular solutions: $\psi(r) \propto r^{-1}$ (S wave)
- cutoff smears contact interaction over range $R \sim \Lambda^{-1}$

$\implies$ need extra factor $\Lambda^{-2}$ to cancel cutoff dependence from $|\psi(R)|^2 \propto \Lambda^2$ in matrix elements of potential

Other partial waves

- wave functions $\psi(r) \propto r^L$ for small $r$ (assuming no low-energy bound state – regular solution)
- extra factor $\Lambda^{2L}$ needed in potential $\implies$ leading term in $L$-th partial wave of order $Q^2 L$ (Weinberg counting: powers of $Q$ from derivatives of $\delta$-function)
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Three-body systems

Attractive: 3 bosons or 3 distinct fermions in unitary limit (triton)

- naive dimensional analysis $\rightarrow$ leading contact term of order $Q^3$
- as hyperradius $R \rightarrow 0$ wave functions behave like

$$\psi(R) \propto R^{-2 \pm i s_0} \quad s_0 \approx 1.006 \quad \text{[Efimov (1971)]}$$

$\rightarrow$ leading three-body force promoted to order $Q^{-1}$
- marginal perturbation associated with limit cycle of RG
  [Bedaque, Hammer and van Kolck (1999)]
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Repulsive: 1 distinct and 2 identical fermions in unitary limit
(alkali atoms or neutrons)
  • hyperradial wave functions $\psi(R) \propto R^{-2+2.1662}$
    $\rightarrow$ leading three-body force of noninteger order $Q^{3.3324}$
How do pion-exchange forces affect the power counting?

Treat $\lambda_{NN}$ as low-energy scale $\rightarrow$ iterate OPE

Central OPE (spin-singlet waves)

- $1/r$ singularity – not enough to alter power-law forms of wave functions at small $r$, even if iterated
- $L \geq 1$ waves: weak scattering $\rightarrow$ Weinberg power counting
- $^1S_0$: similar to expansion around unitary fixed point
  $\rightarrow$ KSW-like power counting
Tensor OPE (spin-triplet waves)

- $1/r^3$ singularity
- but higher partial waves protected by centrifugal barrier
- above critical momentum waves resolve singularity
  → OPE not perturbative
- $L \geq 3$: $p_c \gtrsim 2$ GeV → Weinberg counting OK
- $L \leq 2$: $p_c \lesssim 3m_\pi$ → new counting needed

[Nogga, Timmermans and van Kolck (2005)]
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  [Nogga, Timmermans and van Kolck (2005)]
- wave functions $\psi(r) \propto r^{-1/4}$ multiplied by either sine or exponential function of $1/\sqrt{\lambda_{NN}r}$
  $\rightarrow$ leading contact interaction of order $Q^{-1/2}$ in P, D waves
  (very weakly irrelevant)
Three-body forces

Two-pion exchange

• purely long-range interactions
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One-pion exchange ("$c_D$")

• contains two-body contact vertices like
  $$(N^\dagger N)(N^\dagger \sigma \tau N) \cdot \nabla \pi$$
• promoted in same way as contact interactions for $L \leq 2$
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Contact interaction ("$c_E$")

- counting still not known:
  need to solve 3-body problem with $1/r^3$ potentials [L Platter]
- expect to be promoted → order $Q^d$, $-1 < d < 3$?
So, how should we build an effective Hamiltonian?

<table>
<thead>
<tr>
<th>Order</th>
<th>NN</th>
<th>NNN</th>
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<tbody>
<tr>
<td>$Q^{-1}$</td>
<td>$^1S_0$, $^3S_1$ $C_0$'s, LO OPE</td>
<td></td>
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| $Q^{-1/2}$ | $^3P_J$, $^3D_J$ $C_0$'s  
$Q^0$ | $^1S_0$ $C_2$                           |                                          |
| $Q^{1/2}$ | $^3S_1$ $C_2$                           |                                          |
| $Q^{5/4}$ | $^3P_J$, $^3D_J$ $C_2$'s                |                                          |
| $Q^{3/2}$ | $^1S_0$ $C_4$, $^1P_1$ $C_0$, NLO OPE, LO TPE | $^1S_0-^3S_1$ $C_{D0}$ OPE               |
| $Q^{7/4}$ |                                          |                                          |
| $Q^2$   |                                          |                                          |
| $Q^{5/2}$ | $^3S_1$ $C_4$                           | $^3P_J$, $^3D_J$ $C_{D0}$'s OPE          |
| $Q^3$   |                                          | LO 3N TPE                                |
| $Q^?$   |                                          | $C_E$                                    |

- orange terms absent from “N2LO chiral potential” (Weinberg $Q^3$)
- red terms absent from “N3LO” (Weinberg $Q^4$)
- order $Q^{-1}$: have to iterate; order $Q^{-1/2}$: may be better to
Can I iterate my full potential?

Iterating parts of potential and treating others as perturbations – doesn’t fit well with standard few-/many-body methods

Yes, provided you are careful . . .

- resumming subset of higher-order terms
- without the counterterms needed to renormalise them
- dangerous: can alter form of short-distance wave functions and destroy power counting (or, at best, change it)
- but problems don’t arise, provided higher-order terms are small
- general way to ensure this: keep cutoff small, $\Lambda < \Lambda_0$
- introduces artefacts $\propto \left(\frac{Q}{\Lambda}\right)^n \to$ radius of convergence $\Lambda$ not $\Lambda_0 \to$ leaves only a narrow window: $\Lambda$ just below $\Lambda_0$
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Where does all this leave us?

Renormalisation group

→ clear power counting rules for most partial waves
  ● controlled by forms of wave functions as \( r \to 0 \)
  ● in general, not naive dimensional analysis!
  ● two-body couplings directly related to observables
    (DWBA or DW effective-range expansion)
  ● enhancements of other effective operators
    including 3-body forces

Open questions

• counting for 3-body forces in presence of tensor OPE?
• critical momenta for tensor OPE in \( {}^3P_J \), \( {}^3D_J \) waves with \( m_\pi \neq 0 \)?
• same counting for waves where tensor OPE is repulsive?
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