Critical Behavior of heavy quarkonia in medium from QCD sum rules

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Aim of this talk:

1. **Practical procedure of QCDSR calculation**
   1. For introduction: → S.H. Lee’s talk
   2. Borel transformation technique
   3. How to extract spectral properties

2. **Temperature dependence of spectral parameters**
   1. Change across the phase transition: how change of gluon condensates converts into quarkonia?

3. **Construction of the imaginary time correlator**
   1. Reconstructed spectral function
   2. Imaginary time correlator
   3. Comparison with lattice QCD results
Joint CATHIE-INT mini-program Quarkonium in Hot Media: from QCD to Experiment
QCD Sum Rules for Heavy Quarkonium

- **Current-current correlation function**

\[
\Pi^J(q^2) = i \int d^4x e^{iqx} \langle T[j^J(x)j^J(0)] \rangle \quad j^P = i\bar{c}\gamma_5 c, \quad j^V = \bar{c}\gamma_\mu c
\]

\[
\Pi^{P,S}(q^2) = q^2\tilde{\Pi}^J(q^2)
\]

\[
\Pi^{V,A}_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\tilde{\Pi}^J(q^2)
\]

- **Take spacelike momentum**: \( q^2 = -Q^2 < 0 \)

\( \tilde{\Pi} = \tilde{\Pi}^R \)

- **OPE and truncation valid for**:

\( 4m_Q^2 + Q^2 > (\Lambda_{QCD} + aT)^2 \)

- **Temperature effect only through condensates**

- **Meson at rest with respect to medium**: \( q = (\omega,0) \)

- **Longitudinal and transverse polarizations are no longer independent**
Gluon condensates

Smotherer but same amount of change in Full QCD

Full QCD: Cheng et al., ‘08
**Borel Transformation in QCDSR**

**Physical meaning**

- Large $Q^2$ limit + Probing resonance (large $n$)
- Suppression of high energy part of $\rho(s)$

$$
\mathcal{M}(M^2) = \lim_{Q^2/n \to M^2, \quad n, Q^2 \to \infty} \frac{(Q^2)^{n+1} \pi}{n! \left( -\frac{d}{dQ^2} \right)^n} \Pi(Q^2)
$$

$$
= \int_{4m_Q^2}^{\infty} ds e^{-s/M^2} \text{Im}\Pi(s) \quad \leftrightarrow \quad \text{Dispersion relation}
$$

$$
\rho^{\text{ph}}(s) = \frac{1}{\pi} \frac{f \Gamma \sqrt{s}}{(s - m^2)^2 + s \Gamma^2} + \frac{1}{\pi} \text{Im}\Pi^{\text{pert}}(s) \theta(s - s_0)
$$

$$
- \frac{1}{\partial(1/M^2)} \left[ \mathcal{M}(M^2) - \mathcal{M}^{\text{cont}}(M^2) \right] = \frac{\int_{4m_Q^2}^{\infty} ds \ s e^{-s/M^2} \rho^{\text{pole}}(s)}{\int_{4m_Q^2}^{\infty} ds e^{-s/M^2} \rho^{\text{pole}}(s)}
$$

$$
(= m^2 \quad \text{if} \quad \rho^{\text{pole}}(s) = f \delta(s - m^2))
$$
**OPE side : temperature dependence**

\[
\mathcal{M}(M^2) = e^{-4m_Q^2/M^2} \pi A(M^2) \left[ 1 + \alpha_s(M^2)a(M^2) + \phi_b b(M^2) + \phi_c c(M^2) \right]
\]

- **Behavior of each OPE terms**
  - **Scalar** \((b\phi_b)\)
    - Negative at low \(T\)
      - \(G_0 > 0\)
      - \(b < 0\)
    - Increase with \(T\)
  - **Twist-2** \((c\phi_c)\)
    - Always positive
    - Increase with \(T\)

- **Parameter**
  - \(\alpha_s(8m_c^2) = 0.21\),
  - \(m_c(p^2 = -m_c^2) = 1.26\text{GeV}\),
  - \(G_0^{\text{vac}} = (0.35 \text{ GeV})^4\)
Sum rule constraints

From the OPE...

$$\text{OPE} = \int ds \, e^{-s/M^2}$$

If other quantities are fixed:
- $m^2$: decrease
- $\Gamma$: Increase
- $s_0$: decrease
- $f$: Increase
Borel Window : where QCDSR works well

Borel Window : $M^2$ range such that...

- Criterion 1 – OPE convergence in the Window
  - Power correction is small enough
    \[
    \frac{\text{max(gluon condensates)}}{\text{total OPE}} \leq 0.3
    \]

- Criterion 2 – Pole should dominate
  \[
  \frac{M_{\text{cont}}(M^2)}{M(M^2)} < 0.3
  \]

- Criterion 3 – Mass should not depend on $M^2$, or must have local minimum/maximum
How to choose the best solution?

- Searching for \((m, \Gamma, s_0)\) giving the flattest Borel curve

\[
\chi^2 \equiv \frac{1}{M_{\text{max}}^2 - M_{\text{min}}^2} \int_{M_{\text{min}}^2}^{M_{\text{max}}^2} dM^2 (m^2(M^2) - m^2(M_0^2))^2
\]

\[
\left. \frac{d m(M^2)}{d M^2} \right|_{M^2 = M_0^2} = 0
\]

- Caveats

  - Solution is not unique!
    - Many combination can give similarly flat curve!
    - Need to fix either \(\Gamma\) or \(s_0\)

  - Changing \(s_0 \rightarrow M_{\text{max}}^2\) changes
    - This method seems to work only when either \(M_{\text{min}}^2\) or \(M_{\text{max}}^2\) is fixed
    - Useful for determine \(\Gamma\) and \(m\) at a fixed \(T\) and \(s_0\)
Systematics of Borel curve modification

\[ M^2_{\text{min}}, M^2, M^2_{\text{max}} \]

- \( T=0, \Gamma=0, \) larger \( s_0 \)
- \( T=0, \Gamma=0, \) moderate \( s_0 \)
- \( T=0, \Gamma > 0, \) moderate \( s_0 \)

Favored
Systematics of Borel curve modification

- $T=0$, $\Gamma=0$, moderate $s_0$
- $T\sim T_c$, $\Gamma=0$, moderate $s_0$
- $T\sim T_c$, $\Gamma>0$, moderate $s_0$
- $T\sim T_c$, $\Gamma=0$, smaller $s_0$ (favored)
Systematics of Borel curve modification

**Equations and Text:**

1. \( T \sim T_c, \Gamma = 0, \text{moderate } s_0 \)
2. \( T > 1.05 T_c, \Gamma = 0, \text{moderate } s_0 \)
3. \( T > 1.05 T_c, \Gamma = 0, \text{smaller } s_0 \)
4. \( T > 1.05 T_c, \Gamma > 0, \text{moderate } s_0 \)

**Graphical Representation:**

- Minimum exists only if width or threshold changes.
- Beyond \( T_{\text{onset}} \), width is necessary for obtaining minimum.

**Legend:**

- \( M^2_{\text{min}} \) and \( M^2_{\text{max}} \) indicate minimum and maximum values.

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**Footnotes:**

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Results for Charmonia
Results: S-wave states

$\eta_c$ starts to broaden earlier?

$T_{\text{onset}} = 1.07T_c$

$T_{\text{onset}} = 1.04T_c$
Results: P-wave states

\[ T_{\text{onset}} = 1.05T_c \]
Indication from the results

Constraints among $m$, $\Gamma$ and $s_0$
- One needs to fix $m$ or $s_0$ (not $\Gamma$)

Effect of temperature: as $T$ increases,

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Mass should decrease around $T_c$
- 2nd order Stark effect (Lee-Morita PRD75)

Caveat:
- $T>T_{\text{onset}}$, higher dim. condensates will be important
Combined with 2nd Order Stark effect

ALL quantities change abruptly around $T_c$!!!
Combined with 2nd Order Stark effect

ALL quantities change abruptly around $T_c$!!!
Comparison with lattice QCD?

- MEM spectral density: too coarse, especially above $T_c$
  - Spectral modification from QCDSR: $O(100\text{MeV})$

- More accurate quantity: Imaginary time correlator
  \[ G(\tau, T) = \int d^3x \langle J(\tau, x) j^\dagger(0) \rangle \]
  \[ = D^>(-i\tau, T) \]

-Dispersion relation
  \[ G(\tau, T) = \int_0^\infty d\omega \frac{\cosh \left[ \omega \left( \tau - \frac{1}{2T} \right) \right]}{\sinh \left( \frac{\omega}{2T} \right)} \rho(\omega, T) \]

We can put the phenomenological side used in QCDSR.
Modeled spectral density

Parameters from the result combined with Stark effect
Imaginary time correlator: results

Looking at medium modification

\[ \frac{G(\tau, T)}{G_{\text{rec}}(\tau, T)}, \quad G_{\text{rec}}(\tau, T) = \int_0^\infty d\omega \frac{\cosh \left[ \omega \left( \tau - \frac{1}{2T} \right) \right]}{\sinh \left( \frac{\omega}{2T} \right)} \rho(\omega, T = 0) \]

J/\psi, T=0.87T_c=257 MeV

Deviation at \( \tau > 0.2 \text{fm} \)

Below \( T_c \): zero-mode contribution should be different

Lattice: Jakovac et al., PRD75,014506 ('07)
**Imaginary time correlator: results**

- **Looking at medium modification**

  \[
  \frac{G(\tau, T)}{G_{\text{rec}}(\tau, T)}, \quad G_{\text{rec}}(\tau, T) = \int_0^\infty d\omega \frac{\cosh [\omega (\tau - \frac{1}{2T})]}{\sinh (\frac{\omega}{2T})} \rho(\omega, T = 0)
  \]

- **J/ψ, T=0.87T_c=257 MeV**

  - Neglecting zero mode
  - Agree with lattice when continuum threshold does not change (no mass change in this case)
Imaginary time correlator: model study

- Varying only one parameter

![Graphs showing imaginary time correlator behavior under different parameters and normalizations.]

- Normalized at $s_0^{1/2} = 3.48$ GeV
- Normalized at $m = 2.993$ GeV
- Normalized at $\Gamma = 2$ MeV
- $\rho_{\text{peak}} = \frac{f_0 \Gamma}{m}$
- $f = 1.2$, $f = 1.1$, $f = 0.9$, $f = 0.8$
Imaginary time correlator: $J/\psi$ above $T_c$

$J/\psi$, $T=1.07T_c=316$ MeV

Agreement in lower $s_0$ cases at $\tau < 0.2$

Zero mode contribution seems overestimated?
**Imaginary time correlator:** $J/\psi$ above $T_c$

\[ G(\tau, T)/G_{\text{rec}}(\tau, T) \]

- Lattice, $\beta=6.1$
- $s_0^{1/2}=3.7$ GeV
- $s_0^{1/2}=3.6$ GeV
- $s_0^{1/2}=3.5$ GeV
- $s_0^{1/2}=3.4$ GeV
- $s_0^{1/2}=3.3$ GeV
- $s_0^{1/2}=3.2$ GeV
- $s_0^{1/2}=3.1$ GeV
- $s_0^{1/2}=3.05$ GeV

- $J/\psi$, $T=1.07T_c=316$ MeV

Agreement in lower $s_0$ cases at $\tau < 0.2$

Zero mode contribution seems overestimated?

- $\sqrt{s_0} = 3.2$ GeV
- $\delta m = -120$ MeV
- $\Gamma = 64$ MeV

→ Also consistent with Stark effect

w/o zero mode, $G/G_{\text{rec}} \sim 1$ is possible with a mixture of pole and continuum modification
Imaginary time correlator: $J/\psi$ above $T_c$

$J/\psi$, $T = 1.09T_c = 322$ MeV

Lattice data: finest spacing

Agreement becomes worse

Zero mode contribution seems overestimated?

Caveat: $T = 1.09T_c$ is above the onset of the width

Improvement needed

- Correct zero mode → More modification needed
- Check zero mode through $\chi$ states
Imaginary time correlator: $J/\psi$ above $T_c$

- $J/\psi$, $T=1.09T_c=322$ MeV
- Lattice data: finest spacing
- Agreement becomes worse
- Zero mode contribution seems overestimated?
- Caveat: $T=1.09T_c$ is above the onset of the width
- Improvement needed

- Correct zero mode $\rightarrow$ More modification needed
- Check zero mode through $\chi$ states
Imaginary time correlator: $\chi_c$ above $T_c$

- Zero mode contribution dominates
- Agree with lattice data but not sensitive to detailed spectral change
Imaginary time correlator: $\chi_c$ above $T_c$

- Zero mode contribution dominates; in free gas approximation, it increases as $T$ increases.
- Larger than lattice: change from $1.07T_c$?
Imaginary time correlator: $\eta_c$ above $T_c$

- No zero mode in PS channel
- Larger than lattice: different from $J/\psi$
**Discussion**

**Too simple continuum?**

- In QCDSR, continuum is well suppressed
- How about in the correlator?

\[ G(\tau, T) = \int_0^\infty d\omega K(\tau, T) \rho(\omega) \]

\[ \mathcal{M}(M^2) = 2 \int_0^\infty \frac{d\omega}{\omega} e^{-\omega^2/M^2} \rho(\omega) \]

ITC might be sensitive to what QCDSR is not.

Note: we have imposed all higher states effects on continuum threshold.

Explicit excited states (2s) will enhance \( G_{\text{rec}} \) in the model then reduce \( G/G_{\text{rec}} \)

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Jun 18, 2009

Joint CATHIE-INT mini-program Quarkonium in Hot Media: from QCD to Experiment
Relative contribution from each $\rho_i(\omega)$

- In $G_{rec}$, temperature effect is small at $T=1.07 T_c$
- Continuum dominates the correlator
- Zero mode $< 20\%$ in V channel
Relative contribution from each $\rho_i(\omega)$

- In $G_{rec}$, temperature effect is small at $T=1.07T_c$
- Continuum dominates the correlator
- Zero mode contribution dominates at large $\tau$
Relative contribution from each $\rho_i(\omega)$

- In $G_{rec}$, temperature effect is small at $T=1.07T_c$
- Continuum dominates the correlator at $\tau < 0.2\text{fm}$
- Zero mode dominates at $\tau > 0.25\text{fm}$
Results for Bottomonia:

- Condensates contribution to OPE is much smaller:
  \[(m_b/m_c)^4 \sim 100\]
- \(m_b = 4.15 \text{ GeV}, \ \alpha_s(8m_b^2)=0.12\)
- \(M_{\text{min}}^2: 5\% \text{ dim.4 contribution}\)
Spectral modification above $T_c$

$T_{\text{onset}} = 2.09 T_c$

$T_{\text{onset}} = 1.96 T_c$
Combined with 2\textsuperscript{nd} order Stark effect

Both seem to exist at 2T_c
Spectral modification above $T_c$

$T_{\text{onset}} = 1.52T_c$

$T_{\text{onset}} = 1.46T_c$
Imaginary time correlator: $\Upsilon$

- Agree with lattice data at $T=1.15T_c$ : $\delta m = -25\text{MeV}$, $\Gamma=0$,
- Deviation at $T=1.54T_c$
Imaginary time correlator: $\eta_b$

- Agree with lattice data at $T=1.15T_c$: $\delta m = -18\text{MeV}$, $\Gamma=0$,
- Deviation at $T=1.54T_c$
Imaginary time correlator: $\chi_{b0}$

- Agree with coarser lattice, not with finer one
- Deviation at $T=1.54T_c$
Summary and Outlook

- **QCDSR** gives constraints on the spectral modification
  - Instability of the Borel curve indicates onset of broadening
- **QCDSR**: translates change of condensates into pole
  - Charmonium: sudden changes across $T_c$
  - Bottomonium: moderate changes above $T_c$
- **$G/G_{\text{rec}}$**: more sensitive to continuum
  - QCDSR-constrained modification is not inconsistent with lattice $G/G_{\text{rec}}$
    - Decreasing threshold, mass, and overlap constant / Increasing width
    - Agreement might be accidental
  - For clarification, we need more realistic continuum model including excited states and thermal effects, which do NOT affect QCDSR
A trial...

- Including $\psi'$ at $T=0$ (using exp. data from PDG)
  - QCDSR re-fit: only 0.3% change of mass and 10% increase of continuum threshold
  - Imaginary time correlator at $T=0$: 3% change

![Graphs showing the effect of $\psi'$ on the correlator at $T=0$.]
Backup
OPE in the heavy quark systems

**Truncation at the leading order**

\[ q^2 = I + G^2 \]

\[ (G^n\text{term}) \sim \int_0^1 \frac{F(q^2, x)dx}{[m_h^2 - x(1-x)q^2]^n} \langle G^n \rangle \]

if \( 4m_h^2 - q^2 \gg \Lambda_{QCD}^2 \), higher dimensional ones can be neglected

**Introducing temperature** (Hatsuda-Koike-Lee, ’93)

- Low enough / small change from vacuum value

\[ 4m_h^2 - q^2 \gg \Lambda_{QCD}^2, T^2 \]

Temperature dependence only through operators
Mathematical aspects (Bertlmann, '82)

- **Limit of hypergeometric function**

\[
F \left( n, b, n + c; \frac{\xi}{1 + \xi} \right) \to \frac{n^b}{\Gamma(c)} \int_0^\infty ds \frac{1}{s^c} \frac{1}{(\omega + s)^{-b}}
\]

\[
\equiv n^b G(b, c, \omega)
\]

- **Whittaker function**

**Results**

\[
\mathcal{M}(M^2) = e^{-\frac{4m_Q^2}{M^2}} \pi A(M^2) \left[ 1 + \alpha_s(M^2) a(M^2) + \phi_b b(M^2) + \phi_c c(M^2) \right]
\]

\[
\pi A^V(\omega) = \frac{3}{16\sqrt{\pi}} \frac{4m_h^2}{\omega} G \left( \frac{1}{2}, \frac{5}{2}, \omega \right)
\]

\[
a^V(\omega) = \frac{4}{3\sqrt{\pi} G \left( \frac{1}{2}, \frac{5}{2}, \omega \right)} \left[ \pi - c_1 G(1, 2, \omega) + \frac{1}{3} c_2 G(2, 3, \omega) \right] - c_2 - \frac{4 \ln 2}{\pi} G \left( \frac{1}{2}, \frac{3}{2}, \omega \right)
\]

\[
b^V(\omega) = -\frac{\omega^2 G \left( -\frac{1}{2}, \frac{3}{2}, \omega \right)}{2 G \left( \frac{1}{2}, \frac{5}{2}, \omega \right)}
\]

\[
c^V(\omega) = b^V(\omega) - \frac{2}{3} \omega^3 G \left( \frac{3}{2}, \frac{3}{2}, \omega \right)
\]

\[
= b^V(\omega) - \frac{2}{3} \omega^3 G \left( \frac{3}{2}, \frac{3}{2}, \omega \right)
\]
QCD sum rules

\[ \Pi(Q^2) = \int_0^\infty ds \frac{\rho(s)}{s + Q^2} \]

Hadronic spectral density (what we want to know)

OPE at large \( Q^2 \)

Matching to obtain \( m^2 \) and \( s_0 \)

\[ \simeq C_{\text{pert}}(Q^2) + \sum_i C_{q_i\bar{q}_i} \langle m_i \bar{q}_i q_i \rangle + C_G \left\langle \frac{\alpha_s}{\pi} G_{\mu\nu} G^{\mu\nu} \right\rangle + \cdots \]

R.H.S is just a SUM of spectral density, but there are many ways to optimize it!
**Gluon condensates in medium**

- **Appearance of the twist-2 gluon operator**

\[
\begin{align*}
\langle \alpha_s G^2 \rangle & = C_G^0 \frac{\alpha_s}{\pi} G^2 \\
\frac{1}{q^2} \langle \alpha_s G^{\mu\alpha} G^{\nu\alpha} \rangle & = C_G^2 \frac{q^\mu q^\nu}{q^2} \left( \frac{\alpha_s}{\pi} G^{\mu\alpha} G^{\nu\alpha} \right)
\end{align*}
\]

- **Relation to thermodynamic quantities**

  - **Trace anomaly + traceless/symmetric term**

\[
\langle \frac{\alpha_s}{\pi} G^{\mu\alpha} G^{\nu\alpha} \rangle = \left( u_\mu u_\nu - \frac{1}{4} g_{\mu\nu} \right) G_2(T) + \frac{1}{4} g_{\mu\nu} G_0(T)
\]

  - **Energy-momentum tensor**

\[
\langle T^\mu_\mu \rangle = \left( \frac{\beta(g)}{2g} G^{\alpha\mu} G^{\mu\alpha} \right) = \varepsilon - 3p
\]

\[
T^{\mu\nu} = -G^{\alpha\mu\alpha} G^{\alpha\nu} = (\varepsilon + p) u^\mu u^\nu
\]
Temperature dependence from lattice QCD

\[ G_0(T) = G_0^{\text{vac}} - \frac{8}{11}(\varepsilon - 3p), \quad G_2(T) = -\frac{\alpha_s(T)}{\pi}(\varepsilon + p) \]

- Rapid change around \( T_c \)
- \( G_2 \) quickly approaches to \( T^4 \)
- Max deviation from SB limit at \( 1.1T_c \)

Gluon condensates / \( T^4 \)

\[ \Delta G_0 / T^4 \quad \text{red} \]
\[ G_2 / T^4 \quad \text{blue} \]

\( T/T_c \): 1, 1.5, 2, 2.5, 3

Lattice data: Boyd et al, NPB469,419 ('96)
Examples for $\chi^2$ evaluation ($T=0$)
Examples for $\chi^2$ evaluation ($T=1.0T_c$)
Examples for $\chi^2$ evaluation ($T = 1.07 T_c$)
Imaginary time correlator: Υ

- Less continuum dominant than J/ψ
- T=1.15T_c is similar to T=0 when G/G_{rec}~1
Imaginary time correlator: $\eta_b$

- Agree with lattice data at $T=1.15T_c$: $\delta m = -18\text{MeV}$, $\Gamma=0$,
- Deviation at $T=1.54T_c$
Imaginary time correlator: $\chi_{b0}$

- Agree with coarse lattice but with finer one
- Deviation at $T=1.54T_c$
**Effect of zero-mode**

- **Landau damping contributes the correlator**

  Assuming free thermal quarks in medium

  \[ \rho^{scat}(\omega) = 3\omega \delta(\omega)(c_1 I_1 - c_2 I_2). \]

  \[ G^{scat}(\tau, T) = 3T(c_1 I_1 - c_2 I_2). \]

  We use

  \[ \rho^{QCD\text{SR}}(\omega, T) = \rho^{phen}(\omega^2) + \rho^{scat}(\omega) \]

  - **Caveat:** continuum part may be too simple!
    - We imposed all medium effects on the change of threshold

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