Large $N$ Limits & Equivalences

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Based on work done in collaboration with Pavel Kovtun and Mithat Ünsal


“New frontiers in large $N$ gauge theories,” INT, Feb 3, 2009
Simplifications as $N \rightarrow \infty$

- Topological diagrammatic expansion
  \[ \Rightarrow \text{planar diagrams dominate} \]
- Factorization: $\langle AB \rangle \rightarrow \langle A \rangle \langle B \rangle$
- Closed loop equations:
  \[ W_{\Gamma} = a_{\Gamma}^{\Gamma'} W_{\Gamma'} + b_{\Gamma}^{\Gamma''} W_{\Gamma'} W_{\Gamma''} \]
- Vanishing meson, glueball widths
- Scattering amplitudes $\sim (N)^2 \times \# \text{particles}$
- Baryons $\sim$ solitons
- Volume independence

Why?
Large $N$ limit = Thermodynamic limit

Phase transitions, spontaneous symmetry breaking, coexisting equilibrium states:

- Possible in large volume limit.

  Cluster decomposition:
  \[ \langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/V) \]
  
  Diagnostic of extremal (pure) equilibrium state in large volume limit

- Possible in large $N$ limit.

  Factorization:
  \[ \langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/N) \]
  
  Diagnostic of extremal (pure) state in large $N$ limit

*Assuming finite correlation length
Large $N$ limit = Classical limit

$N \to \infty$: quantum dynamics $\to$ classical dynamics

- Large $N$ coherent states $\{|u\rangle\}$ $\sim$ classical phase space

- Quantum operators $\to$ classical observables

  $$a(u) = \lim_{N \to \infty} \langle u | A | u \rangle$$

- Vanishing overlaps: $\langle u | u' \rangle \sim e^{-N^2 f(u,u')}$

  $$\lim_{N \to \infty} \langle u | A B | u \rangle = \lim_{N \to \infty} \langle u | A | u \rangle \langle u | B | u \rangle = a(u) b(u)$$

- Classical action:

  $$S_{cl} = \lim_{N \to \infty} \frac{1}{N^{2}} \int dt \langle u | i \partial_t - \hat{H} | u \rangle$$

  $\Rightarrow$ ground state properties, spectrum, scattering amplitudes, ...

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Coherent states

Produced by action of “coherence group” $\mathcal{G}$: $|u\rangle = U|0\rangle$, $U \in \mathcal{G}$

Coherence group $\mathcal{G}$ generated by invariant “coordinates” & “momenta”:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>Point particles</td>
<td>${q_\alpha, p_\beta}$</td>
</tr>
<tr>
<td>$N$-component vectors</td>
<td>${\vec{\phi}<em>\alpha \cdot \vec{\phi}</em>\beta, \vec{\pi}<em>\alpha \cdot \vec{\phi}</em>\beta}$</td>
</tr>
<tr>
<td>$U(N)$ gauge theory</td>
<td>${\text{tr } U_\Gamma, \text{tr } E_\alpha U_\Gamma}$</td>
</tr>
</tbody>
</table>
Reality check

Above $d=2$:

- Can’t sum planar diagrams
- Can’t solve $N=\infty$ loop equations
- Can’t analytically minimize $S_{\text{cl}}$ on infinite-dimensional phase space
- Difficult to formulate useful finite-dimensional truncation

But...

- Can use loop equations, or coherent state dynamics, to compare large $N$ limits of differing theories
Large $N$ equivalences

Differing finite $N$ gauge theories can have identical* large $N$ limits:

Gauge group independence
$U(N)$ vs. $O(N)$ vs. $Sp(N)$

Volume independence

Orbifold projections

Orientifold projections

*With important caveats...

Lovelace 1982

Eguchi & Kawai 1982, Bhanot, Heller & Neuberger 1982, Gonzalez-Arroyo & Okawa 1983, ...


Armoni, Shifman & Veneziano 2003, ...
Orbifold projections

“Parent” theory:

Choose discrete symmetry \( P \subset (\text{gauge} \otimes \text{spacetime} \otimes \text{flavor}) \) operators, states invariant under \( P \equiv \text{“neutral”}, \text{non-invariant} \equiv \text{“non-neutral” or “twisted”} \)

Eliminate degrees of freedom not invariant under \( P \)

“Daughter” theory:

May have “emergent” non-gauge symmetry \( Q \), not present in parent operators, states invariant under \( Q \equiv \text{“neutral”}, \text{non-invariant} \equiv \text{“non-neutral” or “twisted”} \)

Operator mapping:

\[ \{ \text{neutral single-trace operators} \}_{\text{parent}} \leftrightarrow \{ \text{neutral single-trace operators} \}_{\text{daughter}} \]
Neutral sector equivalence

“Invertible” projections ➞ non-perturbative equivalence of dynamics within neutral sectors

non-perturbative equivalence of leading large $N$ behavior

of connected correlators of neutral operators provided symmetries defining neutral sector not spontaneously broken

directly relate leading large $N$ behavior of free energy,

as well as spectrum, partial decay widths & scattering amplitudes of neutral glueballs & mesons
$Z_2$ projection "duality" web

<table>
<thead>
<tr>
<th>Projection</th>
<th>Gauge Group</th>
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</thead>
<tbody>
<tr>
<td>$K = \begin{bmatrix} 1 &amp; \end{bmatrix}$</td>
<td>$U(2N) \rightarrow U(N)^2$</td>
</tr>
<tr>
<td>$C = \text{charge conj.}$</td>
<td>$U(2N) \rightarrow O(2N)$</td>
</tr>
<tr>
<td>$J = \begin{bmatrix} -1 &amp; 1 \end{bmatrix}$</td>
<td>$O(2N) \rightarrow U(N)$</td>
</tr>
<tr>
<td>$C \times J$</td>
<td>$U(2N) \rightarrow Sp(2N)$</td>
</tr>
</tbody>
</table>

Diagram:

- $O(4N) \rightarrow [O(2N)]^2 \rightarrow O(2N) \rightarrow [O(N)]^2 \rightarrow [U(2N)]^2 \rightarrow U(2N) \rightarrow [U(N)]^2 \rightarrow U(N) \rightarrow [Sp(2N)] \rightarrow Sp(2N) \rightarrow [Sp(N)]^2 \rightarrow Sp(4N)$
## Some specific examples

<table>
<thead>
<tr>
<th>Projection</th>
<th>Parent Theory $\rightarrow$ Daughter Theory</th>
<th>Emergent Daughter Theory Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>orbifold</td>
<td>$U(2N) \text{ SYM} \rightarrow U(N)^2 \text{ YM w. bifund. ferm.}$</td>
<td>$U(N)_1 \leftrightarrow U(N)^2_2$</td>
</tr>
<tr>
<td>orientifold</td>
<td>$SO(2N) \text{ SYM}$</td>
<td>$U(N) \text{ SYM}$</td>
</tr>
<tr>
<td></td>
<td>$J$</td>
<td>$J (-1)^F$</td>
</tr>
<tr>
<td></td>
<td>$U(N) \text{ QCD(AS)}$</td>
<td>charge conjugation</td>
</tr>
<tr>
<td>volume reduction</td>
<td>$U(N) \text{ YM on } (KL)^d \rightarrow U(N) \text{ YM on } (L)^d$</td>
<td>$(Z_N)^d$ center</td>
</tr>
</tbody>
</table>

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Symmetry realization engineering

Example: $Z_N$ center symmetry in compactified $U(N)$ YM

“deconfinement” = failure of original Eguchi-Kawai proposal

“Fixes”:

quenched EK

twisted EK

$YM \rightarrow QCD(\text{Adj})$ ✓

$YM \rightarrow \text{center-stabilized YM}$ ✓

$$S_{\text{center-stabilized}}^{YM} = S_{\text{Wilson}}^{YM} + \sum_{n=1}^{[N/2]} c_n \left| \text{tr} L^n \right|^2, \quad c_n \text{ sufficiently positive}$$

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Numerical utility?

Reproduce $C$-even properties of large volume, large-$N$ QCD(AS) using single-site $SU(N)$ matrix model with light adjoint fermions

Reproduce properties of large volume, large-$N$ YM using single-site matrix model with center-stabilizing terms
Some open questions

- Does $U(N)^2$ Yang-Mills with sufficiently light bifundamental fermions spontaneously break the gauge group interchange symmetry?
- Does QCD(AS) in large volume ever spontaneously break charge conjugation symmetry?
- How large must $N$ be in QCD(Adj), or center-stabilized YM, for accurate volume independence down to a single-site?
- Is numerical simulation cost of single-site center-stabilized YM manageable (relative to large volume simulations) or prohibitive?
- Can large $N$ equivalences improve understanding of phenomenologically interesting models of new strong dynamics sectors?
- Can one formulate accurate finite-dimensional truncations of dynamics on infinite dimensional large $N$ phase space of gauge theories?