PHASE TRANSITION, DUALITY AND CORRESPONDENCES
OF A SIMPLE GAUGE SYSTEM AT INFINITE N

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1 YANG-MILLS QUANTUM MECHANICS

\[ QCD|_{V=\infty} \rightarrow QCD|_{V=0} \]

\[ H = \frac{1}{2} p^i p_i^a + \frac{g^2}{4} \varepsilon_{abc} \varepsilon_{ade} x^i_b x^i_c x^i_d x^i_e + \frac{ig}{2} \varepsilon_{abc} \psi^a \Gamma^k \psi_b x^k_c, \]

\[ i = 1, \ldots, D - 1 \quad a = 1, \ldots, N^2 - 1. \]

⇒ Bjorken (’79) – femto-universe
⇒ Lüscher (’83) – lattice small volume expansion
⇒ Banks, Fischler, Shenker, Susskind (’97) – M-theory
The spectrum is calculable!

⇒ States, the Fock space:

\[ |n\rangle = \frac{a^n}{\sqrt{n!}} |0\rangle, \quad n \text{ no of quanta} \]

\[ \langle m|H|n\rangle \Rightarrow E_m, \quad \psi_m(x) \quad (1) \]

⇒ The cutoff

\[ n \leq n_{max} \Rightarrow E_m(n_{max}), \quad n_{max} \rightarrow \infty \quad (2) \]
Figure 1: The spectrum of the SU(2) supersymmetric Yang-Mills quantum mechanics in 3+1 dimensions (with M. Campostrini)

M. Campostrini, M. Trzetrzelewski, J. Kotanski, P. Korcył
P. van Baal, R. Janik
2 THE LARGE N LIMIT (in Hamiltonian formulation)

Only single trace states contribute at large N.
Only single trace operators are relevant
Matrix elements calculable analytically

hep-th/0512301, 0603045, 0607198, 0609210, 0610172
mat-ph/0603082 with E. Onofri

E. Onofri et al., M. Beccaria, P. Korcyl
3 ONE SUPERSYMMETRIC HAMILTONIAN

\[ Q = \sqrt{2} Tr[f a^\dagger (1 + g a^\dagger)], \]
\[ Q^\dagger = \sqrt{2} Tr[1 + g a] \]
\[ H = \{Q, Q^\dagger\} = H_B + H_F. \]

\[ H_B = Tr[a^\dagger a + g(a^\dagger a^2 + a^\dagger a + a^2) + g^2 a^\dagger a^2]. \]
\[ H_F = Tr[f^\dagger f + g(f^\dagger f(a^\dagger + a) + f^\dagger (a^\dagger + a)f) \]
\[ + g^2 (f^\dagger af a^\dagger + f^\dagger aa^\dagger f + f^\dagger fa^\dagger a + f^\dagger af)] \]

LARGE N MATRIX ELEMENTS OF \( H \)

F=0 \( |0, n\rangle = Tr[a^\dagger^n]|0\rangle / \sqrt{N^n} \)

\[ \langle 0, n|H|0, n \rangle = (1 + \lambda(1 - \delta_{n1}))n, \]
\[ \langle 0, n + 1|H|0, n \rangle = \langle 0, n|H|0, n + 1 \rangle = \sqrt{\lambda} \sqrt{n(n + 1)}. \]

F=1

\[ \langle 1, n|H|1, n \rangle = (1 + \lambda)(n + 1) + \lambda, \]
\[ \langle 1, n + 1|H|1, n \rangle = \langle 1, n|H_2|1, n + 1 \rangle = \sqrt{\lambda}(2 + n). \]
THE SPECTRUM

Figure 2: First 10 energy levels of $H$ in $F=0$ and $F=1$ sectors at $\lambda = 0.5$
Supersymmetry is unbroken except by the cutoff.

Good test of the planar calculus.

Well defined system for all values of ’t Hooft coupling.

BUT

*All* levels collapse at $\lambda_c = 1$!
Figure 3: The cutoff dependence of the spectra of $H$, in the F=0 sector for a range of $\lambda$'s
THE PHASE TRANSITION

The critical slowing down
Any finite number of levels collapses at $\lambda_c = 1$ - the spectrum looses its energy gap - it becomes continuous.
Second ground state with $E = 0$ appears in the strong coupling phase.
Rearrangement of supermultiplets.
Witten index has a discontinuity at $\lambda_c$.
The strong - weak duality.
ANALYTIC SOLUTION

CONSTRUCTION OF THE SECOND GROUND STATE

\[ b \equiv \sqrt{\lambda} \]  \hspace{1cm} (3)

\[ |0\rangle_2 = \mathcal{N} \sum_{n=1}^{\infty} \left( \frac{-1}{b} \right)^n \frac{1}{\sqrt{n}} |0, n\rangle, \quad \mathcal{N} = \frac{1}{\sqrt{\log \frac{\lambda}{\lambda-1}}}. \]  \hspace{1cm} (4)

STRONG/WEAK DUALITY

F=0

\[ b \left( E_n^{(F=0)}(1/b) - \frac{1}{b^2} \right) = \frac{1}{b} \left( E_{n+1}^{(F=0)}(b) - b^2 \right). \]  \hspace{1cm} (5)

F=1

\[ b \left( E_n^{(F=1)}(1/b) - \frac{1}{b^2} \right) = \frac{1}{b} \left( E_n^{(F=1)}(b) - b^2 \right). \]
3.1 SPECTRUM AND EIGENSTATES

The planar basis

\[ |0, n \rangle = \frac{1}{\mathcal{N}_n} \text{Tr}[a^{in}] |0 \rangle \]

A non-orthonormal (but useful) basis:

\[ |B_n \rangle = \sqrt{n} |n \rangle + b\sqrt{n + 1} |n + 1 \rangle. \]

The generating function \( f(x) \) for the expansion of the eigenstates \( |\psi > \) into the \( |B_n \rangle \) basis.

\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \leftrightarrow \quad |\psi \rangle = \sum_{n=0}^{\infty} c_n |B_n \rangle \]

The \( H \psi = E \psi \) \( \Rightarrow \)

\[ w(x)f'(x) + xf(x) - \epsilon f(x) = bf(0) + f'(0), \]
\[ w(x) = (x + b)(x + 1/b), \quad E = b(\epsilon + b) \]
The solution

\[
    f(x) = \frac{1}{\alpha} \frac{1}{x + 1/b} F(1, \alpha; 1 + \alpha; \frac{x + b}{x + 1/b}), \quad b < 1,
\]

\[
    f(x) = \frac{1}{1 - \alpha} \frac{1}{x + b} F(1, 1 - \alpha; 2 - \alpha; \frac{x + 1/b}{x + b}), \quad b > 1,
\]

\[
    E = \alpha(b^2 - 1)
\]

The quantization condition

\[
    f(0) = 0 \Rightarrow E_n \text{ reproduces the numerical eigenvalues of } \langle m|H|n \rangle
\]

One more check: set \( \alpha = 0 \) in the \( b > 1 \) solution.

\[
    f_0(x) = \frac{1}{1 + bx} \log \frac{b + x}{b - 1/b}, \quad b > 1,
\]

Generates the second vacuum state as it should.

\[
    \text{do this for } b < 1 - \text{there is no such state at weak coupling!}
\]
$F=2,3$

States with $F$ fermions are labeled by $F$ bosonic occupation numbers (configurations).

$$|n\rangle = |n_1, n_2, \ldots, n_F\rangle = \frac{1}{\mathcal{N}_{\{n\}}} Tr(a^{\dagger n_1} f^{\dagger} a^{\dagger n_2} f^{\dagger} \ldots a^{\dagger n_F} f^{\dagger}) |0\rangle$$

Cyclic shifts give the same state

Pauli principle $\rightarrow$ some configurations are not allowed, e.g.

$$\{n, n\}, \quad \text{or} \quad \{2, 1, 1, 2, 1, 1\}$$

Degeneracy factors
Figure 4: **Low lying bosonic and fermionic levels in the first four fermionic sectors.**

**SUPERMULTIPLETS**

supermultiplets OK

F=(0 - 1) accommodate complete representations of SUSY, but F=(2 - 3) do not.

Richer structure than in 0/1, e.g. not equidistant levels.
REARRANGEMENT OF F=2 AND F=3 SUPERPARTNERS

The phase transition is there, as in 0/1 sectors.

Supermultiplets rearrange across the phase transition point.

Two new vacua appear in the strong coupling phase!

The exact construction of both vacua.
Figure 5: **Rearrangement of the** $F = 2$ (red) and $F = 3$ (black) **levels while passing through the critical coupling** $\lambda_c = 1$. 
Figure 6: First five supersymmetry fractions.

SUPERSYMMETRY FRACTIONS

\[ q_{mn} \equiv \sqrt{\frac{2}{E_m + E_n}} \langle F + 1, E_m | Q^\dag | F, E_n \rangle \]  \hspace{1cm} (7)
RESTRICTED WITTEN INDEX

\[ W(T, \lambda) = \sum_i (-1)^{F_i} e^{-TE_i} \]

No good when supermultiplets are incomplete (if no SUSY).
New definition - ”analytic continuation” into the critical region.

\[ W_R(T, \lambda) = \sum_i \left( e^{-TE_i} - e^{-T\bar{E}_i} \right), \quad \bar{E}_i = \frac{\sum_f E_f |q_{fi}|^2}{\sum_f |q_{fi}|^2} \]
Figure 7: Behaviour of the restricted Witten index, at $T = 6$, around the phase transition.
THE STRONG COUPLING LIMIT

\[ H_{\text{strong}} = \lim_{\lambda \to \infty} \frac{1}{\lambda} H = \]
\[ Tr(f^\dagger f) + \frac{1}{N} [Tr(a^\dagger a^2) + Tr(a^\dagger f^\dagger a f) + Tr(f^\dagger a^\dagger f a)]. \] (8)

It conserves both \( F \) and \( B = n_1 + n_2 + \ldots + n_F \).

Still has exact supersymmetry.

\( H_{\text{strong}} \) is the finite matrix in each \( (F, B) \) sector (c.f. a map of all sectors).

The SUSY vacua are only in the sectors with even \( F \) and \( (F, B = F \pm 1) \) – the magic staircase
\begin{center}
\begin{tabular}{c|cccccccccc}
11 & 1 & 1 & 6 & 26 & 91 & \ldots & \ldots & \ldots & \ldots & \ldots & 16796 \\
10 & 1 & 1 & 5 & 22 & 73 & 201 & 497 & 1144 & \ldots & \ldots & \ldots \\
9  & 1 & 1 & 5 & 19 & 55 & 143 & 335 & 715 & \textbf{1430} & \ldots & \textbf{4862} \\
8  & 1 & 1 & 4 & 15 & 42 & 99 & 212 & 429 & 809 & 1430 & 2424 \\
7  & 1 & 1 & 4 & 12 & 30 & 66 & \textbf{132} & 247 & 429 & 715 & 1144 \\
6  & 1 & 1 & 3 & 10 & 22 & 42 & 76 & 132 & 217 & 335 & 497 \\
5  & 1 & 1 & 3 & 7 & \textbf{14} & 26 & \textbf{42} & 66 & 99 & 143 & 201 \\
4  & 1 & 1 & 2 & 5 & 9 & 14 & 20 & 30 & 43 & 55 & 70 \\
3  & 1 & 1 & \textbf{2} & 4 & \textbf{5} & 7 & 10 & 12 & 15 & 19 & 22 \\
2  & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 & 5 & 5 & 5 \\
1  & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0  & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
\textbf{B} & \textbf{F} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{tabular}
\end{center}

Table 1: Sizes of gauge invariant bases in the (F,B) sectors.

The magic staircase $\Rightarrow$ there are always two SUSY vacua at finite $\lambda$ (in the strong coupling phase).
4 q-BOSON GAS

A one dimensional, periodic lattice with length $F$.

A boson at each lattice site $a_i, \ i = 1, ..., F$

The new Hamiltonian

$$H = B + \sum_{i=1}^{F} \delta_{N_i,0} + \sum_{i=1}^{F} b_i b_i^\dagger + b_i^\dagger b_{i-1},$$

(9)

where $N_i = a_i^\dagger a_i$ and $B = n_1 + n_2 + ... + n_F$.

The $b_i^\dagger$ ($b_i$) operators create (annihilate) one quantum without the usual $\sqrt{n}$ factors – assisted transitions.

$$b^\dagger |n\rangle = |n + 1\rangle, \quad b |n\rangle = |n - 1\rangle, \quad b |0\rangle \equiv 0,$$

$$[b, b^\dagger] = \delta_{N,0}$$

(10)

This Hamiltonian conserves $B$.

It is also invariant under lattice shifts $U$.

The spectrum of above $H$,
in the sector with $\lambda_U = -1$, exactly coincides with the spectrum of $H_{strong}$, for even $F$ and any $B$. 

23
q-bosons: the $b$ and $b^\dagger$ c/a operators are defined by

$$b^\dagger = a^\dagger \frac{\sqrt{[N + 1]_q}}{\sqrt{N + 1}}, \quad b = \frac{\sqrt{[N + 1]_q}}{\sqrt{N + 1}} a,$$

$$[x]_q \equiv \frac{1 - q^{-2x}}{1 - q^{-2}},$$

(11)

and satisfy the $q$–deformed algebra of the harmonic oscillator

$$[b, b^\dagger] = q^{-2N}.$$  

(12)

Therefore our SUSY-equivalent system corresponds to $q \rightarrow \infty$.

$q$-Bose gas was considered non-soluble (Bogoliubov) ... until now.
5 THE XXZ MODEL

The one dimensional chain of Heisenberg spins

\[ H^{(\Delta)}_{XXZ} = -\frac{1}{2} \sum_{i=1}^{L} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right) \]

Our planar system, at strong coupling, is equivalent to the XXZ chain with

\( L = F + B, \quad S^z = \sum_{i=1}^{L} s_i^z = F - B, \quad \text{and} \quad \Delta = \pm \frac{1}{2} \)

Riazumov-Stroganov conjecture: for odd \( L \) and \( S^z = \pm 1 \) there exists an eigenstate with known, simple eigenvalue \( E = \frac{3}{4} L \).

\( \Rightarrow \) the R-S states are the SUSY vacua of \( H_{SC} \) !

Even more: there is a hidden supersymmetric structure in the Heisenberg chain.

SUSY relates lattices of different .
6 BETHE ANSATZ

The XXZ model is soluble by the Bethe Ansatz.

The existence of the magic staircase can be proven using BA.

BA can be solved analytically for the first three magic sectors.

Bethe phases for $F=6, B=5 \rightarrow 42 \times 42$

\[ x = \frac{1}{72} \left( 36 + i\sqrt{2}\sqrt{11 + \sqrt{13}(7 + \sqrt{13})} - 6\sqrt{2}\sqrt{6(-3 + \sqrt{13}) + i\sqrt{2}\sqrt{11 + \sqrt{13}(-5 + \sqrt{13})}} \right) \]

\[ y = \frac{1}{72} \left( 36 + i\sqrt{2}\sqrt{11 + \sqrt{13}(7 + \sqrt{13})} + 6\sqrt{2}\sqrt{6(-3 + \sqrt{13}) + i\sqrt{2}\sqrt{11 + \sqrt{13}(-5 + \sqrt{13})}} \right) \]
7 "Understanding" the phase transition in $F = 0, 1$ sectors

Suppose that the matrix

$$\langle 0, n|H|0, n \rangle = (1 + \lambda) n , \quad (14)$$

$$\langle 0, n + 1|H|0, n \rangle = \langle 0, n|H|0, n + 1 \rangle = \sqrt{\lambda} \sqrt{n(n + 1)} , \quad (15)$$

Resulted in the HO basis, what is corresponding $H(x, p)$?

$$H = \frac{1 + \lambda}{2} (p^2 + x^2 - 1) + \frac{\sqrt{\lambda}}{2} (\sqrt{p^2 + x^2 - 1}(x + ip) + (x - ip)\sqrt{p^2 + x^2 - 1}) \quad (16)$$

The (momentum dependent) potential

$$V(x, p) \simeq \left( \frac{1 + \lambda}{2} \pm \sqrt{\lambda} \right) x^2 , \quad x \rightarrow \pm \infty \quad (17)$$

⇒ At (and only at) $\lambda = 1$ continuous spectrum of incoming and reflected waves!
8 FROM $N=3,4,5$ TO INFINITY

Figure 8: Lowest eigenenergy for $N=3,4,5$, and its linear extrapolation to $N=\infty$, together with the planar result