Large $N$ phase transitions under scaling

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Objective: Compute non-perturbatively in QCD at large $N$

Older work
New work
Summary

Colleagues working independently

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Objective: Compute non-perturbatively in QCD at large $N$

1. **Objective:** Compute non-perturbatively in QCD at large $N$
   - HOW?
   - Subjects of interest: Eigenvalues of matrix valued observables.

2. **Older work**
   - Eigenvalue motion under scaling
   - Exact results
   - Numerical studies
   - Extremal eigenvalues

3. **New work**
   - More exact results in 2D QCD
   - Going complex
   - Eigenvalue density, $\rho_N(\theta)$
Objective: *Compute* non-perturbatively in QCD at large $N$

**HOW?**

- Subjects of interest: Eigenvalues of matrix valued observables.

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PLAN

- Find observables undergoing a weak–strong phase transition at $N = \infty$
- Find universality class of large-$N$ transition: determine leading parameters
- Use matched asymptotic expansions to saw together 3 regimes of scales
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DEFINITION OF EIGENVALUES
Zeros of characteristic polynomials.

- QCD in 2, 3, 4 dimensions
  Untraced Wilson loops: \( W = \frac{1}{N} \mathcal{P} e^{i \oint_{C, x \in C} A(y) \cdot dy} \)
  Characteristic polynomial:
  \( Q_N(z) = \langle \det(z - W) \rangle = \prod_{j=1}^{N} (z - z_j(C)) \)
  “Eigenvalues”: \( z_j(C) \).

- Principal chiral model in 2 dimensions
  Untraced correlator: \( W = \frac{1}{N} g(0) g^\dagger(x) \)
  Characteristic polynomial:
  \( Q_N(z) = \langle \det(z - W) \rangle = \prod_{j=1}^{N} (z - z_j(x)) \)
  “Eigenvalues”: \( z_j(x) \)

- \( z_j \)’s are on the unit circle
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THE SPECIFIC OBSERVABLES

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PROBLEM: Quantify motion of $z_j(\lambda)$ under scaling $x \rightarrow \lambda x$
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PHASE TRANSITION AT $N = \infty$

- For short distances, eigenvalues close to 1
- For large distances, eigenvalues spread out uniformly on $|z| = 1$
- At infinite $N$ eigenvalues span a continuum over $|z| = 1$
  - For short distances the eigenvalue density has a gap centered at $z = -1$
  - For large distances the eigenvalue density covers the entire $|z| = 1$
- Under scaling, there is a non-analytic change at some $\lambda = \lambda_c$
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In 2 dimensions dependence on \( \mathcal{C} \Rightarrow \) dependence on area enclosed by \( \mathcal{C}, \mathcal{A} \). The \( \text{'t Hooft coupling} \ g^2N \) has dimensions.

Introduce a dimensionless scaling parameter

\[
\lambda \equiv \tau = A g^2 N \left( 1 + \frac{1}{N} \right) \equiv T \left( 1 + \frac{1}{N} \right)
\]

Exact formula for observable

\[
Q_N(z, \tau) = \sqrt{ \frac{N \tau}{2\pi} } \int_{-\infty}^{\infty} du e^{-\frac{N}{2} \tau u^2} \left[ z - e^{-\tau (u+1/2)} \right]^N
\]
THE UNIVERSAL FUNCTION $\zeta(\xi, \alpha)$

- $N \to \infty \Rightarrow$ scaling regime around $\tau = \tau_c = 4, \, z = -1$

\[
\tau = \frac{4}{1 + \frac{\alpha}{\sqrt{3N}}}; \quad z = -e^{(\frac{4}{3N})^\frac{3}{4}\xi}
\]

- $\alpha$ and $\xi$ are scaling variables
- “Double scaling” limit

\[
\lim_{N \to \infty} \left( \frac{4N}{3} \right)^\frac{1}{4} (-1)^N e^{(\frac{N-1}{8})\tau} (-z)^{-\frac{N}{2}} Q_N(z, \tau) = \int_{-\infty}^{\infty} du e^{-u^4 - \alpha u^2 + \xi u} \equiv \zeta(\xi, \alpha)
\]
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NUMERICAL TESTS

Method

- "Smear" lattice matrices to renormalize
- The renormalized observable is constructed from smeared matrices
  \[ O_N(y, \lambda) = \langle \det(e^{y/2} + e^{-y/2} W(\lambda)) \rangle \]
- The large $N$ universality hypothesis is

\[
\lim_{N \to \infty} \mathcal{N}(N, \lambda) O_N \left( y = \left( \frac{4}{3N^3} \right)^{1/4} \frac{\xi}{a_1}, \lambda = \lambda_c \left[ 1 + \frac{\alpha}{\sqrt{Na_2}} \right] \right) = \zeta(\xi, \alpha)
\]
GETTING $\lambda_c$

- Expand around $y = 0$

  \[ O_N(y, \lambda) = C_0(\lambda, N) + C_1(\lambda, N)y^2 + C_2(\lambda, N)y^4 + \ldots \]

- Define $\Omega(\lambda, N)$, which is independent of $a_1$

  \[ \Omega(\lambda, N) = \frac{C_0(\lambda, N)C_2(\lambda, N)}{C_1^2(\lambda, N)} \]

- Define an effective critical scale $\lambda_c$

  \[ \Omega(\lambda_c(N), N) = \frac{\Gamma^4(\frac{1}{4})}{48\pi^2} = 0.36474 \]
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$\Omega(\alpha)$
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$\Omega(\lambda, N)$

- $\Omega(\lambda, N) = \Omega(\alpha)$, with $\alpha \propto \sqrt{N}(\lambda - \lambda_c)$
- At $N = \infty$, $\Omega(\lambda, N)$ is a step function with $\Omega(0^-) = \frac{1}{6}$, $\Omega(0^+) = \frac{1}{2}$ and $\Omega(0) = 0.36474$
Critical exponents and getting $a_{1,2}$

- Numerically test for the critical exponents 1/2 and 3/4
- $a_2$ is determined from the slope of $\Omega(\lambda, N)$ at $\lambda = \lambda_c$
- $a_1$ is determined from the ratio $C_1/C_0$
- This works in 3D QCD and in 2D $SU(N) \times SU(N)$
- Get consistency, but need $N >> 50$ for an a priori determination of exponents: Not practical
- $N=$integer $\Rightarrow a_1 \equiv 1$ ; consistent with data
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Back to eigenvalues

- $y \sim 0$ corresponds to $Q_N(z, \lambda)$ for $z \sim -1$
- $z \sim -1$ is most sensitive to extremal eigenvalues $-\pi < z_m(\lambda) < 0 < z_M(\lambda) < \pi$
- Critical regime $\lambda \rightarrow \lambda_c$ entered when $-z_m = z_M$ get to the vicinity of $-1$
- Extremal eigenvalues ought to move in a universal way at $\lambda \sim \lambda_c$
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BURGERS’ EQUATION

- Define

\[ q_N(y, \tau) = (-1)^N e^{-\frac{Ny}{2}} e^{-\frac{N\tau}{8}} Q_N(-e^y, \tau) \]

- Take a logarithmic derivative

\[ \phi_N(y, \tau) = -\frac{1}{N} \frac{\partial \log q_N(y, \tau)}{\partial y} \]

- Burgers’ equation

\[ \frac{\partial \phi_N}{\partial \tau} + \phi_N \frac{\partial \phi_N}{\partial y} = \frac{1}{2N} \frac{\partial^2 \phi_N}{\partial y^2}, \quad \phi_N(y, 0) = -\frac{1}{2} \tanh \frac{y}{2} \]
NEW VIEW OF LARGE $N$ UNIVERSALITY

- Solve by method of characteristics at $N = \infty$

\[ \phi(y, 0) = h(y) \Rightarrow \phi(y, \tau) = h(\tau - y\phi(y, \tau)) \]

- Shock at time $\tau^*$, the smallest $\tau > 0$ satisfying

\[ \tau^* = -\frac{1}{(dh/dy)(y^*)} \text{ with } (d^2h/dy^2)(y^*) = 0 \]

- Shock is movable singularity, determined by initial condition

\[ h(y) = -\frac{1}{2} \tanh \frac{y}{2} = -\frac{y}{4} + \frac{y^3}{48} - \ldots \Rightarrow y^* = 0, \tau^* = 4 \]
UNIVERSAL SMOOTHING OF SHOCK

- Viscosity = $\frac{1}{2N}$
  Generic form of dissipative smoothing

- Symmetry
  $y \rightarrow -y$

- Generic initial condition
  Only signs of first two terms in initial condition matter
EQUATIONS OF MOTION
Calogero system for eigenvalues

- Zeros

\[ z_j(\tau) = e^{y_j(\tau)} \]

- Equations of motion

\[
\dot{y}_j = \frac{1}{2N} \sum_{k}^{'} \coth \frac{y_k - y_j}{2} = \frac{1}{N} \sum_{k}^{'} \sum_{n \in \mathbb{Z}} \frac{1}{y_k - y_j + 2n\pi i}
\]

- \( \tau = 0 \) and \( \tau = \infty \)

\[
y_j(0) = 0, \quad y_j(\infty) = \frac{2\pi i}{N} \left( j - \frac{N-1}{2} \right), \quad j = 0, 1, 2, \ldots, N - 1
\]
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Going complex
Eigenvalue density, $\rho_N(\theta)$

ASYMPTOTICS

$\tau \to 0$

\begin{itemize}
\item $\langle \det(z - W) \rangle \sim \det \left[ z - e^{i\sqrt{\frac{\tau}{N}}(a_N + a_N^\dagger)} \right]$
\item $a_N$ is an $N \times N$ matrix
\end{itemize}

$$a_N = \begin{pmatrix}
0 & \sqrt{1} & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & \sqrt{N-1} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}$$

\begin{itemize}
\item $[a_N, a_N^\dagger] = 1_N - NP_N$; \quad $P_N = |N\rangle\langle N|$
\item Non-compact linear spectrum at $N \to \infty$
\end{itemize}
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ASYMPTOTICS

$\tau \to \infty$

- $\tau = \infty$ fixed point; $x_j(\tau) = -i y_j(\tau)$

\[ x_j(\tau = \infty) = \frac{2\pi}{N} \left( j - \frac{N - 1}{2} \right), \quad j = 0, \ldots, N - 1 \]

- Approach to fixed point

\[ \delta x_k(\tau) = \sum_{l=1}^{N-1} \rho_l \sin \left[ \frac{2\pi l}{N} \left( k + \frac{1}{2} \right) \right] e^{-\frac{\tau}{2N} l(N-l)} \]

- with real $\rho_l$ and $\rho_l = \rho_{N-l}$
- exponents are $\sigma_l$, the $l$-string tensions
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$\tau = \tau_c$

- The universal case has an infinite number of computable zeros $y_j$. $z_{m,M}$ correspond to smallest $|y_j|$.
- $z_M \approx \exp[i(\pi - \frac{3.7}{N^{3/4}})]$
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Some regularization scheme might produce

\[ W(C, x) = Pe^{\int_{c, x \in C} [iA \cdot dy + \Phi(y)]|dy|} \]

where, \( \Phi \) is very heavy in the adjoint

As a result eigenvalues are now complex

Diffusion on \( SU(N) \) \( \rightarrow \) diffusion on \( SL(N, C) \)

Same structure: Phase transition at \( t = 4 \).

Convenient map:

\[ z = \frac{u + 1/2}{u - 1/2} \]
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Scatter plot of the eigenvalues of $W$ in the $z$-plane (left) and in the $u$-plane (right) for $t = 3$ (top), $t = 4$ (center), and $t = 5$ (bottom).
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$$\langle \det (z - W) \rangle$$
Singular $\rho_N^{\text{asy}}(\theta)$

- Set $z = e^{i\theta}$ & log-derivative $\Rightarrow \rho_N^{\text{asy}}(\theta)$
- $\rho_N^{\text{asy}}(\theta) \rightarrow \frac{1}{N} \sum_j \delta_{2\pi}(\theta - \theta_j), \quad z_j(\lambda) = e^{i\theta_j}$
- $\rho_N^{\text{asy}}(\theta)$ is singular
<\text{det}\left(\frac{1}{z-W}\right)> \\
\text{Smooth } \rho_N^{\text{sym}}(\theta)

- Set \( z = e^{i\theta} \) & log-derivative \( \Rightarrow \rho_N^{\text{sym}}(\theta) \)
- \( \rho_N^{\text{sym}}(\theta) \) is smooth
- Equation obeyed by \( \rho_N^{\text{sym}} \)

\[
\frac{1}{2N} \frac{\partial^2 \rho_N^{\text{sym}}(\theta, t)}{\partial \theta^2} = \frac{\partial \rho_N^{\text{sym}}(\theta, t)}{\partial t} + \pi \frac{\partial}{\partial \theta} \left[ \rho_N^{\text{sym}}(\theta, t)(H\rho_N^{\text{sym}})(\theta, t) \right]
\]
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$\frac{1}{N} \text{Tr} \left\langle \frac{1}{z-W} \right\rangle$

Oscillatory $\rho^\text{true}_N(\theta)$

- Set $z = e^{i\theta} \Rightarrow \rho^\text{true}_N(\theta)$
- $\rho^\text{true}_N(\theta)$ has $N$ oscillations
  - The peaks of $\rho^\text{true}_N$ are numerically close to $z_j(\lambda)$
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$$\rho_{N}^{\text{sym}}(\theta) = \rho_{N}^{\text{sym}}(-\theta)$$ for increasing $N$
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Eigenvalue density, \( \rho_N(\theta) \)

Compare \( \rho_{50}^{\text{true}} \) with \( \rho_{50}^{\text{sym}} \)

\[ \tau < \tau_c \text{ (left) and } \tau > \tau_c \text{ (right)} \]
Summary

- Large $N$ phase transitions at a critical scale are common.
- Large $N$ universality holds across dimensions/models.
- These are properties of the continuum field theory.

Outlook

- Need to work on our renormalization of zeros $z_j$
- Need to complete asymptotic matching in 2D