Dyson’s instability in the large-N limit

Yannick Meurice
The University of Iowa
yannick-meurice@uiowa.edu

New Frontiers in Large-N Gauge Theories
INT, Seattle, February 6, 2009
With A. Bazavov, A. Denbleyker, D. Du, Y. Liu and A. Velytsky
Content of the talk

• Main questions addressed

• Dyson instability, complex coupling and large fields

• Lattice perturbation theory and density of states (arXiv:0807.0185 [hep-lat], Phys. Rev. D78 054503)

• Sigma models in the large-$N$ limit, in the complex coupling plane

• Open questions for large-$N$ lattice gauge theory
Main Questions Addressed

What is the large order behavior of perturbative series in lattice QCD?

How do truncated series compare with numerical data?

Can we modify the procedure (by carefully considering contributions of the large field configurations) to include the non-pertubative effects?

How can we attack these questions by calculating the density of states (color entropy)

Can we solve or understand better these problems in the large-$N$ limit?
Dyson’s instability

Suppose that a physical quantity in QED can be calculated as a perturbative series $F(e^2) = a_0 + a_1 e^2 + \ldots$.

If we assume that the series has a finite radius of convergence, then, for $e^2$ sufficiently small, we can interpret $F(-|e^2|)$ as the value of this quantity in a fictitious world where same charge particles attract. But in this fictitious world, every physical state is unstable. So, the radius of convergence is zero.

"The argument [...] is lacking in mathematical rigor and in physical precision. It is intended to be suggestive, to serve as a basis for further discussions" (F. J. Dyson, Phys. Rev. 85, 631 (1952))
Complex coupling (Bender, Wu, Zinn-Justin, Parisi, Brezin...)

The validity of Dyson conclusions were confirmed for the anharmonic oscillator, the double-well potential and other models.

Dispersion relations + semi-classical calculations at small negative $\lambda$ or $e^2$ predict the large order behavior of (asymptotic) series.

Theories with stable states at negative coupling can be constructed (Carl Bender et al.)

Large-$N$: some quantities (e. g. , ground state energy, anomalous dimensions ...) have a finite radius of convergence in the ’t Hooft coupling (in the planar approximation). However Dyson instability is invoked by Polyakov (arXiv 0709.2899) in the AdS/CFT context.
Asymptotic series and large fields

\[
\int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2 - \lambda \phi^4} \neq \sum_{0}^{\infty} \frac{(-\lambda)^l}{l!} \int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2} \phi^{4l}
\]

The peak of the integrand of the r.h.s. moves too fast when the order increases. On the other hand, if we introduce a field cutoff, the peak moves outside of the integration range and

\[
\int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2 - \lambda \phi^4} = \sum_{0}^{\infty} \frac{(-\lambda)^l}{l!} \int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2} \phi^{4l} \tag{1}
\]

General expectations: for a finite lattice, the partition function \(Z\) calculated with a field cutoff is convergent and \(\ln(Z)\) has a finite radius of convergence. \(\phi_{max}\) is an optimization parameter fixed using strong coupling, for instance.
Quenched lattice QCD

Lattice gauge theories with a compact group (e.g., Wilson’s lattice QCD) have a build-in large field cutoff: the group elements associated with the links are integrated with $dU_l$ the compact Haar measure. $N$ is the number of colors. UV and large field regularization preserve gauge invariance.

\[
S = \sum_{\text{plaq.}} (1 - (1/N) \text{ReTr}(U_p))
\]

\[
\beta = 2N/g^2
\]

\[
Z = \prod_l \int dU_l e^{-\beta S}
\]

Number of plaquettes: $\mathcal{N}_p \equiv L^D D(D - 1)/2$

Average plaquette: $P(\beta) = (1/\mathcal{N}_p) \left\langle \sum_p (1 - (1/N) \text{ReTr}(U_p)) \right\rangle$
The density of states

$Z(\beta)$ is the Laplace transform of $n(S)$, the density of states

$$Z(\beta) = \int_{0}^{S_{\text{max}}} dS \ n(S) \ e^{-\beta S},$$

with

$$n(S) = \prod_{l} \int dU_{l} \delta(S - \sum_{p} (1 - (1/N) \text{ReTr}(U_{p})))$$

$\ln(n(S))$ is a "color entropy" ($\propto N_{p}$, extensive); $n(S) = e^{N_{p}f(S/N_{p})}$

$S_{\text{max}} = 2N_{p}$ for $SU(2N)$, $\frac{3}{2}N_{p}$ for $SU(3)$; ($N_{p}$ : number of plaquettes)
One plaquette ($SU(2)$)

$$Z(\beta) = \int_0^2 dS n(S) e^{-\beta S} = 2 e^{-\beta} I_1(\beta) / \beta \quad \text{(analytical in the entire } \beta \text{ plane)}$$

$$n(S) = \frac{2}{\pi} \sqrt{S(2-S)} \quad \text{(invariant under } S \to 2-S)$$

The large order of the weak coupling expansion $\beta \to \infty$ is determined by the behavior of $n(S)$ near $S = 2$, itself probed when $\beta \to -\infty$ in agreement with the common wisdom that the large order behavior of weak coupling series can be understood in terms of the behavior at small negative coupling.

$\sqrt{2-S}$ is easy to approximate near $S = 0$ (radius of convergence = 2)

$$Z(\beta) = (\beta \pi)^{-3/2} 2^{1/2} \sum_{l=0}^{\infty} (2/\beta)^{-l} \frac{\Gamma(l+1/2)}{l!(1/2-l)} \int_0^{2\beta} dt e^{-t} t^{l+1/2} \quad \text{is convergent}$$
The crucial step

\[ \int_0^{2\beta} dte^{-t}t^{l+1/2} \simeq \int_0^\infty dte^{-t}t^{l+1/2} + O(e^{-2\beta}) \]

is responsible for the factorial behavior

The peak of the integrand crosses the boundary near order \(2\beta\)

Dropping higher order terms (than order \(\simeq 2\beta\)) agrees with the rule of thumb (minimizing the first contribution dropped)

The non-perturbative part can be fully reconstructed (higher orders + ”tails”, PRD 74 096005)
$L^4$ lattices

The crossing is near order $2\beta N_p$ which explains that up to order 16, no sign of factorial growth is seen on $8^4$ and $24^4$ lattices. However the tail effects may be important for reduced models.

Complex singularities for $|S| < S_{max}$ should explain the behavior of perturbative series at large volume.

Non-perturbative effects should be explainable by the contributions near $S_{max}$ which can be probed at small negative coupling.

$Z$ remains an analytic function of $\beta$ and the strong coupling expansions is dominated by the zeros of the partition function.
Lattice Perturbation Theory ($SU(3)$)

\[ P(1/\beta) = \sum_{m=0}^{10} b_m \beta^{-m} + \ldots . \]

(F. Di Renzo et al. JHEP 10 038, P. Rakow Lat. 05)

Series analysis suggests a singularity: \( P \propto (1/5.74 - 1/\beta)^{1.08} \) (Horsley et al, Rakow, Li and YM)

This means that the coefficients we know grow like \( 5.74^n \) rather than \( n! \)

Not seen in 2d derivative of \( P \) (would requires massless glueballs!)

Solution: complex singularities slightly off the real axis (PRD 73 036006)
Figure 1: \( \ln(b_k) \) for the dilogarithm model (solid line) and the integral model (dashes). The dots up to order 10 are the known values. The two models yields similar coefficients up to order 20. After that, the integral model has the logarithm of its coefficients growing faster than linear.
A $SU(2)$ duality ($g^2 \rightarrow -g^2$ means $S \rightarrow 2N_p - S$)

For cubic lattices with even number of sites in each direction and a gauge group that contains $-1$, it is possible to change $\beta \text{Re} TrU_p$ into $-\beta \text{Re} TrU_p$ by a change of variables $U_l \rightarrow -U_l$ on a set of links such that for any plaquette, exactly one link of the set belongs to that plaquette (Li, YM PRD71 016008). This implies

$$Z(-\beta) = e^{2\beta N_p} Z(\beta)$$

$$n(2N_p - S) = n(S)$$

Thanks to this symmetry, we only need to know $n(S)$ for $0 \leq S \leq N_p$ ($< S > = N_p$ means $< TrU_p > = 0$). Note: this is not a symmetry of the $1^4$ reduced EK model.
For $D = 3$, an example of $\mathcal{L}$ is $\{(A, 0, 0), (0, A, 1), (1, 1, A)\}$ with $A$ arbitrary. It is not difficult to show that there are 8 distinct $\mathcal{L}$. 
Numerical calculation

Figure 2: Results of patching $P_\beta(S)e^{\beta S}$ for $4^4$ and $6^4$ (Phys. Rev. D78 054503).
Volume dependence of the leading log

Figure 3: The difference between \( \ln\left(\frac{n(S)}{N_p}\right) \) (left) and \( \frac{\ln\left(\frac{n(S)}{N_p}\right)}{\ln\left(\frac{S}{N_p}\right)} \) (right) for \( 4^4 \) and \( 6^4 \). Predicted value of the plateau is -0.0013.
Weak and strong coupling expansions

Figure 4: Average plaquette (left) and $\ln(n(S))/N_p$ (right) compared to weak and strong coupling expansions ($x = S/N_p$).
Figure 5: Logarithm of the absolute value of the difference between the numerical data and the weak coupling expansion of $P$ at successive orders (left) and without the zero mode (right).
Figure 6: Comparison of the second and third moment calculated from the density of states and the direct MC result.
Figure 7: Density of states for $U(1)$ on a $4^4$ lattice by multicanonical methods (with A. Bazavov).
Figure 8: Plaquette distribution for $U(1)$ at $\beta=0.978$ (olive), 0.979 (green, symm.), 0.98, and 0.981 (purple), using the density of states for a $4^4$ lattice.
Figure 9: Zeros of Re and Im part of $Z$ for $U(1)$ using the density of states for a $4^4$ lattice. Real part of leading zero is about 0.979. As the volume increases, the zero gets closer.
Linear $O(N)$ $\sigma$-model with a sharp momentum cutoff

$$Z = \int \mathcal{D}\phi e^{-\int d^Dx[(1/2)(\partial\phi)^2 + (1/2)m_B^2\phi^2 + (\lambda^t/N)(\phi^2)^2 + (\eta^t/N^2)(\phi^2)^3]}$$

After introducing auxiliary fields and integrating over the $N$ vector $\vec{\phi}$

$$Z \propto \int_{c-i\infty}^{c+i\infty} dM^2 \int_{0}^{\infty} dX e^{-NV_A}$$

$$\mathcal{A} = (1/2) \int_{|k| \leq 1} \ln(k^2 + M^2) - (1/2)M^2X + U(X)$$

with $U(X) = (1/2)m_B^2X + \lambda^tX^2 + \eta^tX^3 + \ldots$

Every dimensional quantity is expressed in units of a sharp cutoff

(ref: David et al. PRL 53, 2071)
Saddle point (gap equation)

\[ \int_{|k| \leq 1} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + M^2} = X \text{ and } M^2 = 2U'(X) \]

For a quartic potential

\[ \int_{|k| \leq 1} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + M^2} = \left( M^2 - m_B^2 \right) / (4\lambda^t) \]

For \( D=2 \), when \( \lambda^t \) is positive, this equation has 1 solution. When \( \lambda^t \) is negative, this equation may have 2, 1 or no real solution. We call \( \lambda^t_c \) the value where the two real solutions coalesce and disappear if the coupling becomes more negative. (See figure)
Figure 10: Graphical representation of the saddle point equation for $D = 2$ and $m_B^2 = 0.5$, with $\lambda^t$ above, below and very close to $\lambda^t_c$. 
Perturbative solution

Perturbative solution: \( M^2(\lambda^t) = m_b^2 + c_1 \lambda^t + \ldots, \)

\( c_n/c_{n+1} \simeq -0.27(1 + 1.6/n) \) indicates a finite radius of convergence with a square root singularity near \( \lambda_c^t. \)

Numerical calculations with \( m_B^2 = 0.5 \) yield \( \lambda_c^t \simeq -0.27. \)

At large \( M^2, M^2 \propto \sqrt{\lambda^t}, \) so it is plausible that \( M^2(\lambda^t) \) has a square root cut from \(-\infty\) to \( \lambda_c^t. \)

A study of the quadratic fluctuations shows that \( \int \ln(k^2 + M^2), \) induces a local minimum for the "master field" \( X \) when \( \lambda^t \) is not too negative. This is a classical result that is blind to the quantum tunneling (metastability, large field behavior).
Figure 11: Ratios $c_n/c_{n+1}$ and fit.
Nonlinear $O(N)$ sigma model on a cubic lattice

\[ Z = \int \prod_x d^N \phi_x \delta(\vec{\phi}_x \vec{\phi}_x - 1) \, e^{-(1/g_0^2)E[\{\phi\}]} \]

with \( E[\{\phi\}] = -\sum_{x,e}(\vec{\phi}_x \vec{\phi}_{x+e} - 1) \)

We assume a cubic lattice with an even number of sites in each directions and periodic boundary conditions. Under these conditions (as for $SU(2N)$ LGT)

\[ Z[-g_0^2] = e^{2DL^D/g_0^2} Z[g_0^2] \]

This can be seen by changing variable $\phi \rightarrow -\phi$ on sublattices with lattice spacing twice larger and such that they share exactly one site with each link of the original lattice.
Gap equation

$$\prod_{j=1}^{D} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \frac{1}{2(\sum_{j=1}^{D}(1-\cos(k_j))+M^2)} = 1/\lambda^t$$

with $\lambda^t = g_0^2 N$ kept constant as $N$ becomes large.

The saddle point equation is invariant under $\lambda^t \to -\lambda^t$ together with $M^2 \to -M^2 - 4D$. This can be seen by changing variables $k_j \to k_j + \pi$ for all $j$.

For $D = 2$, $\lambda^t \to 0$ when $M^2 \to 0$, $-8, -4 \pm i\epsilon$ with double poles at $(k_1, k_2) = (0, 0), (\pi, \pi), (0, \pi), (0, \pi)$ respectively.
Figure 12: Complex values taken by $\lambda^t$ when $M^2$ varies over the complex plane (here on horizontal and vertical lines in the $M^2$ plane).
Average energy

\[ \mathcal{E} = \langle E \rangle / L^d = (1/2)(\lambda^t - M^2) \]

Note that \(0 \leq \mathcal{E} \leq 2D\) (the range is \(N\)-independent)

At large \(M^2\), \(M^2 \simeq \lambda^t\) so unlike the linear sigma model there is no cut at infinity.

Dispersion relations dominated by four-leaf clover path

Plausible scenario for LGT?
Density of states

\[ n(E) = \int \prod_x d^N \phi_x \delta(\vec{\phi}_x \vec{\phi}_x - 1) \delta(E[\{\phi\}] - E) \]

\[ \delta(E[\{\phi\}] - E) = \int_{K-i\infty}^{K+i\infty} du e^{u(E[\{\phi\}] - E)} \]

Saddle point:

\[ \prod_{j=1}^D \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \frac{1}{2\sum_{j=1}^D (1-\cos(k_j)) + M^2} = u \]

\[ M^2 = \frac{1}{u} - 2\mathcal{E} \]

These equations are equivalent to the previous ones except that now \( \mathcal{E} \) is the independent variable and \( u \) is a function of \( \mathcal{E} \) and plays the role of \( 1/\lambda^t \).
Entropy

\[ f(\mathcal{E}) = \frac{\ln(n(S))}{(N_p N)} \]

\[ = (1/2)M^2 u + u\mathcal{E} \]

\[-(1/2) \log(u) - (1/2) \prod_{j=1}^{D} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \ln(2 \sum_{j=1}^{D} (1 - \cos(k_j)) + M^2) \]

\[ f(\mathcal{E}) = f(2D - \mathcal{E}) \] (using s.p. equations)

small \( \mathcal{E} \), \( f(\mathcal{E}) \approx (1/2) \ln(\mathcal{E}) \)

\( \mathcal{E} \approx D \), \( f(\mathcal{E}) \approx (-1/D)(\mathcal{E} - D)^2 \)
Figure 13: \( f(\mathcal{E}) \) and the leading weak and strong coupling expansions
Perspectives

• $\lambda^t$ expansion sensitive to instability but not metastability

• Clover-leaf dispersion for LGT?

• "Hadamard" expansions (that includes $e^{-l\beta}$ effects) possible for reduced models?

• Large order in weak coupling expansions in LGT dominated by complex singularities of density of states for $0 < |S| < S_{max}$. Non-perturbative effects in tail effects?

• Thanks to the organizers and participants!