What is Orbital Angular Momentum?

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Motivation

- polarized DIS: only $\sim 30\%$ of the proton spin due to quark spins

  ‘spin crisis’ $\rightarrow$ ‘spin puzzle’, because $\Delta \Sigma$ much smaller than the quark model result $\Delta \Sigma = 1$

  quest for the remaining 70%
  - quark orbital angular momentum (OAM)
  - gluon spin
  - gluon OAM

  How are the above quantities defined?

  How can the above quantities be measured
example: angular momentum in QED

consider, for simplicity, first QED without electrons:

\[ \vec{J} = \int d^3r \, \vec{x} \times \left( \vec{E} \times \vec{B} \right) = \int d^3r \, \vec{x} \times \left[ \vec{E} \times \left( \vec{\nabla} \times \vec{A} \right) \right] \]

integrate by parts

\[ \vec{J} = \int d^3r \, \left[ E^j \left( \vec{x} \times \vec{\nabla} \right) A^j \right. \left. + \left( \vec{x} \times \vec{A} \right) \vec{\nabla} \cdot \vec{E} + \vec{E} \times \vec{A} \right] \]

drop 2nd term (eq. of motion \( \vec{\nabla} \cdot \vec{E} = 0 \)), yielding \( \vec{J} = \vec{L} + \vec{S} \) with

\[ \vec{L} = \int d^3r \, E^j \left( \vec{x} \times \vec{\nabla} \right) A^j \quad \vec{S} = \int d^3r \, \vec{E} \times \vec{A} \]

note: \( \vec{L} \) and \( \vec{S} \) not separately gauge invariant
consider now, QED with electrons:

\[
\vec{J}_\gamma = \int d^3r \ \vec{x} \times (\vec{E} \times \vec{B}) = \int d^3r \ \vec{x} \times [\vec{E} \times (\nabla \times \vec{A})]
\]

integrate by parts

\[
\vec{J} = \int d^3r \ [E^j (\vec{x} \times \nabla) A^j + (\vec{x} \times \vec{A}) \vec{\nabla} \cdot \vec{E} + \vec{E} \times \vec{A}]
\]

replace 2nd term (eq. of motion \(\nabla \cdot \vec{E} = e j^0 = e \psi^\dagger \psi\)), yielding

\[
\vec{J}_\gamma = \int d^3r \ [\psi^\dagger \vec{r} \times e \vec{A} \psi + E^j (\vec{x} \times \nabla) A^j + \vec{E} \times \vec{A}]
\]

\(\psi^\dagger \vec{r} \times e \vec{A} \psi\) cancels similar term in electron OAM \(\psi^\dagger \vec{r} \times (\vec{p} - e \vec{A}) \psi\)

decomposing \(\vec{J}_\gamma\) into spin and orbital also shuffles angular momentum from photons to electrons!
total angular momentum of isolated system uniquely defined

ambiguities arise when decomposing \( \vec{J} \) into contributions from different constituents

gauge theories: changing gauge may also shift angular momentum between various degrees of freedom

decomposition of angular momentum in general depends on ‘scheme’ (gauge & quantization scheme)

\[ \text{does not mean that angular momentum decomposition is meaningless, but} \]

one needs to be aware of this ‘scheme’-dependence in the physical interpretation of exp/lattice/model results in terms of spin vs. OAM

and, for example, not mix ‘schemes’, e.t.c.
Outline

- Ji decomposition
- Jaffe decomposition
- recent lattice results (Ji decomposition)
- model/QED illustrations for Ji v. Jaffe
- B.L.T. decomposition
- Chen-Goldman decomposition
The nucleon spin pizza(s)

\[ \Delta G \equiv \sum_q \Delta q \]

\[ \Delta \Sigma \equiv \frac{1}{2} \Delta \Sigma \]

\( \mathcal{L}_q \)

\( J_g \)

\( L_q \)

‘pizza tre stagioni’

‘pizza quattro stagioni’

\[ only \ \frac{1}{2} \Delta \Sigma \equiv \frac{1}{2} \sum_q \Delta q \ \text{common to both decompositions!} \]
Angular Momentum Operator

1. Angular momentum tensor $\mathcal{M}^{\mu \nu \rho} = x^{\mu} T^{\nu \rho} - x^{\nu} T^{\mu \rho}$
2. $\partial_{\rho} \mathcal{M}^{\mu \nu \rho} = 0$

$\vec{J}^{i} = \frac{1}{2} \varepsilon^{i j k} \int d^3 \mathbf{r} M^{j k 0}$ conserved

$$\frac{d}{d t} \vec{J}^{i} = \frac{1}{2} \varepsilon^{i j k} \int d^3 x \partial_{0} M^{j k 0} = \frac{1}{2} \varepsilon^{i j k} \int d^3 x \partial_{l} M^{j k l} = 0$$

- $\mathcal{M}^{\mu \nu \rho}$ contains time derivatives (since $T^{\mu \nu}$ does)
  - use eq. of motion to get rid of time derivatives
  - integrate total derivatives appearing in $T^{0 i}$ by parts
  - yields terms where derivative acts on $x^{i}$ which then ‘disappears’

$\vec{J}^{i}$ usually contains both
  - ‘Extrinsic’ terms, which have the structure ‘$\vec{x} \times$ Operator’, and can be identified with ‘OAM’
  - ‘Intrinsic’ terms, where the factor $\vec{x} \times$ does not appear, and can be identified with ‘spin’
following this general procedure, one finds in QCD

\[ \vec{J} = \int d^3 x \left[ \psi^\dagger \Sigma \psi + \psi^\dagger \vec{x} \times \left( i \vec{\partial} - g \vec{A} \right) \psi + \vec{x} \times \left( \vec{E} \times \vec{B} \right) \right] \]

with \( \Sigma^i = \frac{i}{2} \varepsilon^{ijk} \gamma^j \gamma^k \)

- Ji does not integrate gluon term by parts, nor identify gluon spin/OAM separately
- Ji-decomposition valid for all three components of \( \vec{J} \), but usually only applied to \( \hat{z} \) component, where the quark spin term has a partonic interpretation
  
  (+) all three terms manifestly gauge invariant
  
  (+) DVCS can be used to probe \( \vec{J}_q = \vec{S}_q + \vec{L}_q \)
  
  (-) quark OAM contains interactions
  
  (-) only quark spin has partonic interpretation as a single particle density
Ji-decomposition

\[ \frac{1}{2} = \sum_q J_q + J_g = \sum_q \left( \frac{1}{2} \Delta q + L_q \right) + J_g \]

with \( (P^\mu = (M, 0, 0, 1), S^\mu = (0, 0, 0, 1)) \)

\[ \frac{1}{2} \Delta q = \frac{1}{2} \int d^3x \langle P, S | q^\dagger(\vec{x}) \Sigma^3 q(\vec{x}) | P, S \rangle \quad \Sigma^3 = i\gamma^1\gamma^2 \]

\[ L_q = \int d^3x \langle P, S | q^\dagger(\vec{x}) \left( \vec{x} \times i\vec{D} \right)^3 q(\vec{x}) | P, S \rangle \]

\[ J_g = \int d^3x \langle P, S | \left[ \vec{x} \times \left( \vec{E} \times \vec{B} \right) \right]^3 | P, S \rangle \]

\[ i\vec{D} = i\vec{\partial} - g\vec{A} \]
\[ \vec{J} = \sum_q \frac{1}{2} q^\dagger \vec{\Sigma} q + q^\dagger \left( \vec{r} \times i \vec{D} \right) q + \vec{r} \times \left( \vec{E} \times \vec{B} \right) \]

applies to each vector component of nucleon angular momentum, but Ji-decomposition usually applied only to \( \hat{z} \) component where at least quark spin has parton interpretation as difference between number densities.

\( \Delta q \) from polarized DIS

\( J_q = \frac{1}{2} \Delta q + L_q \) from exp/lattice (GPDs)

\( L_q \) in principle independently defined as matrix elements of \( q^\dagger \left( \vec{r} \times i \vec{D} \right) q \), but in practice easier by subtraction \( L_q = J_q - \frac{1}{2} \Delta q \)

\( J_g \) in principle accessible through gluon GPDs, but in practice easier by subtraction \( J_g = \frac{1}{2} - J_q \)

Ji makes no further decomposition of \( J_g \) into intrinsic (spin) and extrinsic (OAM) piece.
$L_q$ for proton from Ji-relation (lattice)

- lattice QCD $\Rightarrow$ moments of GPDs (LHPC; QCDSF)
- insert in Ji-relation

$$\langle J^i_q \rangle = S^i \int dx \left[ H_q(x, 0) + E_q(x, 0) \right] x.$$

- $L_q^z = J_q^z - \frac{1}{2} \Delta q$

- $L_u, L_d$ both large!
- present calcs. show $L_u + L_d \approx 0$, but
  - disconnected diagrams ..?
  - $m_\pi^2$ extrapolation
  - parton interpret. of $L_q$...

What is Orbital Angular Momentum? – p.12/33
define OAM on a light-like hypersurface rather than a space-like hypersurface

\[ \tilde{J}^3 = \int d^2 x_\perp \int dx^- M^{12+} \]

where \( x^- = \frac{1}{\sqrt{2}} (x^0 - x^-) \) and \( M^{12+} = \frac{1}{\sqrt{2}} (M^{120} + M^{123}) \)

Since \( \partial_\mu M^{12\mu} = 0 \)

\[ \int d^2 x_\perp \int dx^- M^{12+} = \int d^2 x_\perp \int dx^3 M^{120} \]

(compare electrodynamics: \( \vec{\nabla} \cdot \vec{B} = 0 \) ⇒ flux in = flux out)

use eqs. of motion to get rid of ‘time’ (\( \partial_+ \) derivatives) & integrate by parts whenever a total derivative appears in the \( T^{i+} \) part of \( M^{12+} \)
in light-cone framework & light-cone gauge
$A^+ = 0$ one finds for $J^z = \int dx^- d^2r_\perp M^{+xy}$

$$\frac{1}{2} = \frac{1}{2} \Delta \Sigma + \sum_q \mathcal{L}_q + \Delta G + \mathcal{L}_g$$

where $(\gamma^+ = \gamma^0 + \gamma^z)$

$$\mathcal{L}_q = \int d^3r \langle P, S | \bar{q}(\vec{r}) \gamma^+ (\vec{r} \times i\vec{\partial})^z q(\vec{r}) | P, S \rangle$$

$$\Delta G = \varepsilon^{+-ij} \int d^3r \langle P, S | \text{Tr} F^{+i} A^j | P, S \rangle$$

$$\mathcal{L}_g = 2 \int d^3r \langle P, S | \text{Tr} F^{+j} (\vec{x} \times i\vec{\partial})^z A^j | P, S \rangle$$
Jaffe/Manohar decomposition

\[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma + \sum_q \mathcal{L}_q + \Delta G + \mathcal{L}_g \]

- \( \Delta \Sigma = \sum_q \Delta q \) from polarized DIS (or lattice)
- \( \Delta G \) from \( \vec{p} \rightarrow \vec{p} \) or polarized DIS (evolution)
- \( \Delta G \) gauge invariant, but local operator only in light-cone gauge
- \( \int dxx^n \Delta G(x) \) for \( n \geq 1 \) can be described by manifestly gauge inv. local op. (\( \rightarrow \) lattice)
- \( \mathcal{L}_q, \mathcal{L}_g \) independently defined, but
  - no exp. identified to access them
  - not accessible on lattice, since nonlocal except when \( A^+ = 0 \)
- parton net OAM \( \mathcal{L} = \mathcal{L}_g + \sum_q \mathcal{L}_q \) by subtr. \( \mathcal{L} = \frac{1}{2} - \frac{1}{2} \Delta \Sigma - \Delta G \)
- in general, \( \mathcal{L}_q \neq \mathcal{L}_q \quad \mathcal{L}_g + \Delta G \neq J_g \)
- makes no sense to ‘mix’ Ji and JM decompositions, e.g. \( J_g - \Delta G \) has no fundamental connection to OAM
$L_q \neq L_q$

- $L_q$ matrix element of

$$q^+ \left[ \vec{r} \times \left( i \vec{\partial} - g \vec{A} \right) \right] \bar{z} q = \bar{q} \gamma^0 \left[ \vec{r} \times \left( i \vec{\partial} - g \vec{A} \right) \right] \bar{z} q$$

- $L_q^{\bar{z}}$ matrix element of $(\gamma^+ = \gamma^0 + \gamma^{\bar{z}})$

$$\bar{q} \gamma^+ \left[ \vec{r} \times i \vec{\partial} \right] \bar{z} q \bigg|_{A^+ = 0}$$

- (for $p = 0$) matrix element of $\bar{q} \gamma^{\bar{z}} \left[ \vec{r} \times \left( i \vec{\partial} - g \vec{A} \right) \right] \bar{z} q$ vanishes (parity!)

$\leftarrow$ $L_q$ identical to matrix element of $\bar{q} \gamma^+ \left[ \vec{r} \times \left( i \vec{\partial} - g \vec{A} \right) \right] \bar{z} q$ (nucleon at rest)

$\leftarrow$ even in light-cone gauge, $L_q^{\bar{z}}$ and $L_q^{\bar{z}}$ still differ by matrix element of $q^+ \left( \vec{r} \times g \vec{A} \right) \bar{z} q \bigg|_{A^+ = 0} = q^+ (x g A^y - y g A^x) q \bigg|_{A^+ = 0}$

What is Orbital Angular Momentum? – p.16/33
Summary part 1:

- Ji: \( J^z = \frac{1}{2} \Delta \Sigma + \sum_q L_q + J_g \)

- Jaffe: \( J^z = \frac{1}{2} \Delta \Sigma + \sum_q L_q + \Delta G + \mathcal{L}_g \)

\( \Delta G \) can be defined without reference to gauge (and hence gauge invariantly) as the quantity that enters the evolution equations and/or represented by simple (i.e. local) operator only in LC gauge and corresponds to the operator that one would naturally identify with ‘spin’ only in that gauge

- in general \( L_q \neq \mathcal{L}_q \) or \( J_g \neq \Delta G + \mathcal{L}_g \), but

- how significant is the difference between \( L_q \) and \( \mathcal{L}_q \), etc.?
OAM in scalar diquark model

[M.B. + Hikmat Budathoki Chhetri (BC), PRD 79, 071501 (2009)]

- toy model for nucleon where nucleon (mass $M$) splits into quark (mass $m$) and scalar ‘diquark’ (mass $\lambda$)

light-cone wave function for quark-diquark Fock component

$$\psi_{\frac{1}{2}}(x, k_{\perp}) = \left(M + \frac{m}{x}\right) \phi$$

$$\psi_{-\frac{1}{2}} = -\frac{k^1 + ik^2}{x} \phi$$

with $\phi = \frac{c/\sqrt{1-x}}{M^2 - \frac{k^2 + m^2}{x} - \frac{k^2 + \lambda^2}{1-x}}$.

- quark OAM according to JM: $L_q = \int_0^1 dx \int \frac{d^2 k_{\perp}}{16\pi^3} \frac{1}{(1-x)} \left| \psi_{-\frac{1}{2}} \right|^2$

- quark OAM according to Ji: $L_q = \frac{1}{2} \int_0^1 dx \int \left[ q(x) + E(x, 0, 0) \right] - \frac{1}{2} \Delta q$

- (using Lorentz inv. regularization, such as Pauli Villars subtraction) both give identical result, i.e. $L_q = L_q$

- not surprising since scalar diquark model is not a gauge theory

What is Orbital Angular Momentum? – p.18/33
But, even though \( L_q = \mathcal{L}_q \) in this non-gauge theory

\[
\mathcal{L}_q(x) \equiv \int \frac{d^2k_{\perp}}{16\pi^3} (1-x) \left| \psi_{-\frac{1}{2}} \right|^2 \neq \frac{1}{2} \left\{ x[q(x) + E(x, 0, 0)] - \Delta q(x) \right\} \equiv L_q(x)
\]

‘unintegrated Ji-relation’ does not yield \( x \)-distribution of OAM
OAM in QED

- light-cone wave function in $e\gamma$ Fock component
  $$\Psi_{\frac{1}{2}+1}^\uparrow(x, k_\perp) = \sqrt{2} \frac{k^1 - ik^2}{x(1-x)} \phi$$
  $$\Psi_{\frac{1}{2}-1}^\uparrow(x, k_\perp) = -\sqrt{2} \frac{k^1 + ik^2}{1-x} \phi$$
  $$\Psi_{\frac{1}{2}+1}^\downarrow(x, k_\perp) = \sqrt{2} \left( \frac{m}{x} - m \right) \phi$$
  $$\Psi_{\frac{1}{2}-1}^\downarrow(x, k_\perp) = 0$$

- OAM of $e^-$ according to Jaffe/Manohar
  $$L_e = \int_0^1 dx \int d^2k_\perp \left[ (1-x) \left| \Psi_{\frac{1}{2}-1}^\uparrow(x, k_\perp) \right|^2 - \left| \Psi_{\frac{1}{2}+1}^\uparrow(x, k_\perp) \right|^2 \right]$$

- $e^-$ OAM according to Ji
  $$L_e = \frac{1}{2} \int_0^1 dx \; x \left[ q(x) + E(x, 0, 0) \right] - \frac{1}{2} \Delta q$$

  $$L_e \approx L_e + \frac{\alpha}{4\pi} \neq L_e$$

- Likewise, computing $J_\gamma$ from photon GPD, and $\Delta \gamma$ and $L_\gamma$ from light-cone wave functions and defining $\hat{L}_\gamma \equiv J_\gamma - \Delta \gamma$ yields
  $$\hat{L}_\gamma = L_\gamma + \frac{\alpha}{4\pi} \neq L_\gamma$$

- $\frac{\alpha}{4\pi}$ appears to be small, but here $L_e$, $L_e$ are all of $O(\frac{\alpha}{\pi})$
1-loop QCD: \( \mathcal{L}_q - L_q = \frac{\alpha_s}{3\pi} \)

- recall (lattice QCD): \( L_u \approx -0.15 \); \( L_d \approx +0.15 \)
- QCD evolution yields negative correction to \( L_u \) and positive correction to \( L_d \)
- evolution suggested (A.W. Thomas) to explain apparent discrepancy between quark models (low \( Q^2 \)) and lattice results (\( Q^2 \sim 4 GeV^2 \))

- above result suggests that \( \mathcal{L}_u > L_u \) and \( \mathcal{L}_d > L_d \)
- additional contribution (with same sign) from vector potential due to spectators (MB, to be published)

possible that lattice result consistent with \( \mathcal{L}_u > \mathcal{L}_d \)
inclusive $\vec{e}\vec{p}/\vec{p}'\vec{p}$ provide access to
  - quark spin $\frac{1}{2}\Delta q$
  - gluon spin $\Delta G$
  - parton grand total OAM $\mathcal{L} \equiv \mathcal{L}_g + \sum_q \mathcal{L}_q = \frac{1}{2} - \Delta G - \sum_q \Delta q$

DVCS & polarized DIS and/or lattice provide access to
  - quark spin $\frac{1}{2}\Delta q$
  - $J_q$ & $L_q = J_q - \frac{1}{2}\Delta q$
  - $J_g = \frac{1}{2} - \sum_q J_q$

$J_g - \Delta G$ does not yield gluon OAM $\mathcal{L}_g$

$L_q - \mathcal{L}_q = \mathcal{O}(0.1 * \alpha_s)$ for $\mathcal{O}(\alpha_s)$ dressed quark
Chen, Goldman et al.: integrate by parts in \( J_g \) only for term involving \( A_{phys} \), where

\[
A = A_{pure} + A_{phys} \quad \text{with} \quad \nabla \cdot A_{phys} = 0 \quad \nabla \times A_{pure} = 0
\]

\[
\frac{1}{2} = \sum_q J_q + J_g = \sum_q \left( \frac{1}{2} \Delta q + L'_q \right) + S'_g + L'_g \quad \text{with} \quad \Delta q \ \text{as in JM/Ji}
\]

\[
L'_q = \int d^3x \langle P, S | q^\dagger(\vec{x}) \left( \vec{x} \times i\vec{D}_{pure} \right)^3 q(\vec{x}) | P, S \rangle
\]

\[
S'_g = \int d^3x \langle P, S | \left( \vec{E} \times \vec{A}_{phys} \right)^3 | P, S \rangle
\]

\[
L'_g = \int d^3x \langle P, S | E^i \left( \vec{x} \times \vec{\nabla} \right)^3 A^i_{phys} | P, S \rangle
\]

\[
i\vec{D}_{pure} = i\vec{\partial} - g\vec{A}_{pure}
\]

only \( \frac{1}{2} \Delta q \) accessible experimentally
example: angular momentum in QED

consider now, QED with electrons:

$$\vec{J}_\gamma = \int d^3r \vec{x} \times (\vec{E} \times \vec{B}) = \int d^3r \vec{x} \times [\vec{E} \times (\vec{\nabla} \times \vec{A})]$$

integrate by parts

$$\vec{J} = \int d^3r \left[ E^j (\vec{x} \times \vec{\nabla}) A^j + (\vec{x} \times \vec{A}) \vec{\nabla} \cdot \vec{E} + \vec{E} \times \vec{A} \right]$$

replace 2\textsuperscript{nd} term (eq. of motion $\vec{\nabla} \cdot \vec{E} = e\dot{j}^0 = e\psi^\dagger \psi$), yielding

$$\vec{J}_\gamma = \int d^3r \left[ \psi^\dagger \vec{r} \times e\vec{A}\psi + E^j (\vec{x} \times \vec{\nabla}) A^j + \vec{E} \times \vec{A} \right]$$

$\psi^\dagger \vec{r} \times e\vec{A}\psi$ cancels similar term in electron OAM $\psi^\dagger \vec{r} \times (\vec{p} - e\vec{A})\psi$

decomposing $\vec{J}_\gamma$ into spin and orbital also shuffles angular momentum from photons to electrons!
Chen, Goldman et al.: integrate by parts in $J_g$ only for term involving $A_{phys}$, where

$$A = A_{pure} + A_{phys} \quad \text{with} \quad \nabla \cdot A_{phys} = 0 \quad \nabla \times A_{pure} = 0$$
workshop on Orbital Angular Momentum of Partons in Hadrons
ECT* 9-13 November 2009
organizers: M.B. & Gunar Schnell
The Ji-relation (poor man’s derivation)

What distinguishes the Ji-decomposition from other decompositions is the fact that $L_q$ can be constrained by experiment:

$$\langle \vec{J}_q \rangle = \vec{S} \int_{-1}^{1} dx \ x [H_q(x, \xi, 0) + E_q(x, \xi, 0)]$$

(nucleon at rest; $\vec{S}$ is nucleon spin)

$\leftrightarrow \quad L \hat{z} = J \hat{z} - \frac{1}{2} \Delta q$

derivation (MB-version):

- consider nucleon state that is an eigenstate under rotation about the $\hat{x}$-axis (e.g. nucleon polarized in $\hat{x}$ direction with $\vec{p} = 0$ (wave packet if necessary)

- for such a state, $\langle T_{q00}^{00} y \rangle = 0 = \langle T_{q00}^{zz} y \rangle$ and $\langle T_{q0y}^{00} z \rangle = -\langle T_{q0z}^{00} y \rangle$

$\leftrightarrow \quad \langle T_{q}^{++} y \rangle = \langle T_{q0y}^{00} z - T_{q0z}^{00} y \rangle = \langle J_{q}^{x} \rangle$

$\leftrightarrow$ relate $2^{nd}$ moment of $\perp$ flavor dipole moment to $J_{q}^{x}$
derivation (MB-version):
- consider nucleon state that is an eigenstate under rotation about the $\hat{x}$-axis (e.g. nucleon polarized in $\hat{x}$ direction with $\vec{p} = 0$ (wave packet if necessary)
- for such a state, $\langle T_{q}^{00} y \rangle = 0 = \langle T_{q}^{zz} y \rangle$ and $\langle T_{q}^{0y} z \rangle = -\langle T_{q}^{0z} y \rangle$
- $\langle T_{q}^{++} y \rangle = \langle T_{q}^{0y} z - T_{q}^{0z} y \rangle = \langle J_{q}^{x} \rangle$
- relate 2\textsuperscript{nd} moment of $\perp$ flavor dipole moment to $J_{q}^{x}$
- effect sum of two effects:
  - $\langle T_{q}^{++} y \rangle$ for a point-like transversely polarized nucleon
  - $\langle T_{q}^{++} y \rangle$ for a quark relative to the center of momentum of a transversely polarized nucleon
- 2\textsuperscript{nd} moment of $\perp$ flavor dipole moment for point-like nucleon

$$\psi = \left( \frac{f(r)}{\vec{\sigma} \cdot \vec{p}} \frac{f(r)}{E + m} \right) \chi \quad \text{with} \quad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
The Ji-relation (poor man’s derivation)

- derivation (MB-version):
  - \( T^0_z q = i\bar{q} \left( \gamma^0 \partial^z + \gamma^z \partial^0 \right) q \)
  - since \( \psi \dagger \partial_z \psi \) is even under \( y \rightarrow -y \), \( i\bar{q} \gamma^0 \partial^z q \) does not contribute to \( \langle T^0_z y \rangle \)
  - using \( i\partial_0 \psi = E\psi \), one finds

\[
\langle T^0_z b_y \rangle = E \int d^3r \psi \dagger \gamma^0 \gamma^z \psi y = E \int d^3r \psi \dagger \begin{pmatrix} 0 & \sigma^z \\ \sigma^z & 0 \end{pmatrix} \psi y \\
= \frac{2E}{E + M} \int d^3r \chi \dagger \sigma^z \chi f(r)(-i)\partial^y f(r)y = \frac{E}{E + M} \int d^3r f^2(r)
\]

- consider nucleon state with \( \vec{p} = 0 \), i.e. \( E = M \) & \( \int d^3r f^2(r) = 1 \)

\( \leftarrow 2^{nd} \) moment of \( \perp \) flavor dipole moment \( \langle T^{++}_q y \rangle = \langle T^0_z b_y \rangle = \frac{1}{2} \)

\( \leftarrow \) ‘overall shift’ of nucleon COM yields contribution \( \frac{1}{2} \int dx x H_q(x, 0, 0) \) to \( \langle T^{++}_q y \rangle \)
The Ji-relation (poor man’s derivation)

- spherically symmetric wave packet for Dirac particle with $J_x = \frac{1}{2}$ centered around the origin has $\perp$ center of momentum $\frac{1}{M} \langle T_++ b_y \rangle$ not at origin, but at $\frac{1}{2M}$!

- consistent with

$$\frac{1}{2} = \langle J_x \rangle = \langle (T^0 z b_y - T^0 y b_z) \rangle = 2 \langle T^0 z b_y \rangle = \langle T^{++} b_y \rangle$$

- ‘overall shift of $\perp$ COM yields’ $\langle T_++ b_y \rangle = \frac{1}{2} \int dx \ x H_q(x, 0, 0)$

- intrinsic distortion adds $\frac{1}{2} \int dx \ x E_q(x, 0, 0)$ to that
Transversely Deformed Distributions and $E(x, 0, -\Delta^2_\perp)$


- So far: only unpolarized (or long. pol.) nucleon! In general ($\xi = 0$):

  \[ \int \frac{dx^-_4}{4\pi} e^{ip^+} x^- x \langle P + \Delta, \uparrow | \bar{q}(0) \gamma^+ q(x^-) | P, \uparrow \rangle = H(x, 0, -\Delta^2_\perp) \]

  \[ \int \frac{dx^-_4}{4\pi} e^{ip^+} x^- x \langle P + \Delta, \uparrow | \bar{q}(0) \gamma^+ q(x^-) | P, \downarrow \rangle = -\frac{\Delta_x i \Delta_y}{2M} E(x, 0, -\Delta^2_\perp). \]

- Consider nucleon polarized in $x$ direction (in IMF)

  $|X\rangle \equiv |p^+, R_\perp = 0_\perp, \uparrow \rangle + |p^+, R_\perp = 0_\perp, \downarrow \rangle$.

  $\rightarrow$ unpolarized quark distribution for this state:

  \[ q(x, b_\perp) = \mathcal{H}(x, b_\perp) - \frac{1}{2M} \frac{\partial}{\partial b_y} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} E(x, 0, -\Delta^2_\perp) e^{-i b_\perp \cdot \Delta_\perp} \]

- Physics: $j^+ = j^0 + j^3$, and left-right asymmetry from $j^3$!

[X.Ji, PRL 91, 062001 (2003)]
What is Orbital Angular Momentum? – p.32/33
The Ji-relation (poor man’s derivation)

- ‘overall shift of ⊥ COM yields $\langle T_\perp^{++} b_y \rangle = \frac{1}{2} \int dx \, x H_q(x, 0, 0)$
- intrinsic distortion adds $\frac{1}{2} \int dx \, x E_q(x, 0, 0)$ to that
- Ji relation

$$J^x_q = \frac{1}{2} \int dx \, x [H_q(x, 0, 0) + E_q(x, 0, 0)]$$

- rotational invariance: should apply to each vector component, but parton interpretation (transverse shift) only for ⊥ pol. nucleon