Random matrices and chaos in the nuclear shell model

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Content

Correlations between states with different quantum numbers

Geometric properties of the shell model

Fitting of the effective interaction
Preponderance of spin-0 ground states in the two-body random ensemble

Aims of these studies: Understand the generic properties of a the nucleus as a quantum man-body system with rotational symmetry (and isospin symmetry).
Level repulsion in nuclei and the nuclear shell model

Nuclear data

FIG. 8. The NNS distribution for the nuclear data ensemble (histogram) and the GOE prediction (solid line). From Haq et al., 1983.

Nuclear shell model


→ Two-body random ensemble (TBRE) [French & Wong 1970, Bohigas & Flores 1971]
Two-body random ensemble


• Hamiltonian matrix

\[ H_{\mu \nu} (J) = \sum_{\alpha} v_\alpha C_{\mu \nu} (J, \alpha) \]

Hamiltonian matrix with many-body spin J

Sum over all TBME

Two-body matrix element
Random Gaussian variable

Matrix of TBME in many-fermion system at total spin J

• TBRE is not accessible analytically. Numerical studies.
Structure of the Hamiltonian matrix

Hamiltonian:

\[ H_{\mu\nu}(J) = \sum_{\alpha} \nu_{\alpha} C_{\mu\nu}(J, \alpha) \]

TBME:

\( \nu_{\alpha} \)

Geometry (spin/isospin coupling):

\( C_{\mu\nu}(J, \alpha) \)

Example: J=T=0 matrix for \(^{28}\text{Si}\). Built from 63 TBME

25.2 \( \pm \) 4.0 of the 28 block-diagonal C’s contribute to a given matrix element: Thorough mixing.

7.2 \( \pm \) 1.4 of the 22 off-diagonal C’s which change partition by one (green) contribute to a given matrix element.

2.0 \( \pm \) 0.9 of the 13 off-diagonal C’s which change partition by two (blue) contribute to a given matrix element. Altogether thorough mixing.
Correlations between states with different quantum numbers

- To what degree are states with different quantum numbers (spin/isospin, mass number) correlated?
- This question arises when fitting an effective interaction. (To what degree can the Hamiltonian matrix be reconstructed from levels and matrix elements?) Example:

- from the “observation” that the TBRE tends to produce spin-0 ground states (for even systems)

- Choose more suitable decomposition of Hamiltonian
Useful decomposition of shell-model Hamiltonians

• Original formulation:

\[ H_{\mu \nu}(J) = \sum_{\alpha} v_\alpha C_{\mu \nu}(J, \alpha) \]

• More useful description in terms of other linear combinations

\[ H_{\mu \nu}(J) = \sum_{\alpha=1} w_\alpha(J) s_\alpha(J) B_{\mu \nu}(J, \alpha) \]

• B-matrices are ‘orthonormalized’

\[
\frac{1}{d(J)} \text{Trace}[B(J, \alpha)B(J, \beta)] = \delta_{\alpha\beta}
\]

• Can be obtained from diagonalization of a (small) overlap matrix

\[ S_{\alpha\beta}(J) = d^{-1}(J) \text{Trace}[C(J, \alpha)C(J, \beta)] \]

• Eigenvalues \( s_\alpha \) of overlap matrix \( S \) particularly interesting! They are operator norms.
Eigenvalues

Key features:

• One dominant operator → monopole operator
• Many operators (linear combinations) have small norm; very difficult to determine by fit to data!
• Some operators have zero norm! $\mathcal{J}^2 - J(J + 1)$
• Spin-0 norms are slightly enhanced → dominance of spin-0 ground states
• Inherent geometrical properties of the shell model

Regularity from random interactions

Motivation: Shell models with random two-body interactions really like spin-0 ground states [Johnson, Bertsch & Dean 1998].

Similar behavior found for bosons [Bijker & Frank 2000] and electrons [Jacquod & Stone 2000].

Understood for solvable model. (It's geometry!) [Chau Huu-Tai et al. 2002; Zhao & Arima 2001].

Quantitative predictions: [Zhao, Arima, and Yoshinaga 2002, 2003; Papenbrock and Weidenmüller 2004, Yoshinaga, Arima, and Zhao 2006]

Recent reviews: [Zelevinsky & Volya 2004; Zhao, Arima & Yoshinaga 2004].

Correlations between spectral widths account for this observation
### Puzzle

<table>
<thead>
<tr>
<th>(N)</th>
<th>(\Omega)</th>
<th>(\text{nucleus})</th>
<th>(J = 0, T = T_z)</th>
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</thead>
<tbody>
<tr>
<td>6</td>
<td>12</td>
<td>(^{22}\text{O})</td>
<td>76%</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>(^{46}\text{Ca})</td>
<td>75%</td>
</tr>
<tr>
<td>(N = 4, Z = 4)</td>
<td>12</td>
<td>(^{24}\text{Mg})</td>
<td>66%</td>
</tr>
</tbody>
</table>


![Graph showing percentage of states with spin J and percentage of ground states with spin J.](image)

6 fermions in a single \(j\)-shell.
Explaining the puzzle: line of arguments

1. Start from spin-dependent width (width = Tr H^2)
2. Spectral radius (ground state) is proportional to width. (Fluctuations of proportionality factor r_J can be neglected.)
3. Width fluctuations and width-width correlations important. Ensemble averages provide no insight.
4. Fluctuations and correlations can be understood (width is quadratic form).

Spectral width: \[ \sigma_J^2 = \frac{1}{d_J} \text{Tr} H^2 = \sum_{\alpha,\beta} v_\alpha v_\beta S_{\alpha\beta} = \sum_{\alpha} w_\alpha^2 s_\alpha^2 \]

Width related to spectral radius: \[ R_J \approx r_J \sigma_J \]

Def.: \[ R_J = \max_i |E_i| \] for spectrum \( E_i \) at spin \( J \).
The simple explanation works quite well


FIG. 2. (Color online) TBRE indicates $P(I)$ for 1000 random interactions with $j = 11/2$ and $n = 4$, Emp-pred indicates the empirical method of Ref. [7], and Width-pred indicates the method proposed in this paper.
$P_J(\sigma) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} dt e^{it\sigma^2} \prod_{\alpha=1}^a \frac{e^{-\frac{i}{2} \arctan 2ts^2}}{(1 + 4s^4 t^2)^{1/4}}$
Correlation between widths belonging to different J

\[ p(J, J') = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty dt \, \frac{\sin\left(\frac{1}{2} \sum_{\alpha} \arctan 2tq_{\alpha}\right)}{t \Pi_{\alpha} (1 + 4q_{\alpha}^2 t^2)^{1/4}} \]

\( q_{\alpha} \): eigenvalues of matrix \( S_{\alpha\beta}(J) - S_{\alpha\beta}(J') \)

Correlation (overlap) between eigenvectors of overlap matrices belonging to spins J and J’

→ Dominant linear combination depends weakly on many-body spin J
Correlations between spectra of different spins

No such correlations exist in the GOE, but in the TBRE they do.

FIG. 11. (Color online) Correlations between the \( J = 0, T = 0 \) states and the \( J = 1, T = 0 \) states of \(^{24}\text{Mg}\) (from 400 realizations of the ensemble). (Left panel) Average of the product of the two level densities. (Middle panel) Product of the averages of the two level densities. (Right panel) The correlator.

FIG. 12. (Color online) Same as described in the legend to Fig. 11 but for the \( J = 0, T = 0 \) states of \(^{24}\text{Mg}\) and of \(^{22}\text{Ne}\) (from 400 realizations of the ensemble).
Can such correlations be seen experimentally?

- Data are not sufficiently reliable (complete) → use shell-model spectra
- Correlations between spacings of $J = 0$ and $J = 2$ states (or for $J = 1/2$ and $J = 5/2$ states)
- Two-body interaction (Brown-Wildenthal)
- Ensemble of nuclei: $^{20-24}\text{Ne}$, $^{22-24}\text{Na}$, $^{24-26}\text{Mg}$, $^{26}\text{Al}$, $^{30,32,34}\text{Si}$, $^{34}\text{P}$, $^{36}\text{Ar}$.

\[ \overline{\Delta J \Delta J'} \quad \overline{\Delta J \cdot \Delta J'} \quad \overline{\Delta J \Delta J'} - \overline{\Delta J \cdot \Delta J'} \]

**FIG. 13.** (Color online) Level spacing correlations for 17 $sd$-shell nuclei between spins $J = 0$ and $J = 2$ ($J = 1/2$ and $J = 5/2$) for even-$A$ (odd-$A$) nuclei. (Left) Average of product of spacings; (Middle) product of average spacings; (Right) correlation.

Correlations decay within ~4 spacings
Abundance of even parity states

Model: $m$ fermions on $l_1$ orbitals with positive and $l_2$ orbitals with negative parity

\[
E_{\text{ground}}(\pm) = n \text{Tr}(H \mathcal{P}_\pm) - r_{\pm} \sigma_{\pm}
\]

\[
\sigma_{\pm}^2 = n \text{Tr}(H^2 \mathcal{P}_\pm)
\]

Numerical results:
- dimension sets abundance
- simple estimate works well

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Analytical results for dilute systems in large spaces:
Equal abundance of even and odd-parity ground states

Summary

Correlations in the TBRE

- Analysis of suitable linear combinations of TBME
- Explains difficulties in fitting effective interactions
- Understanding of spin-0 dominance in TBRE
- Correlations between spectra of different quantum numbers