The SM prediction of the muon g-2

Massimo Passera
INFN Padova

Symposium on Muon Physics in the LHC Era
INT - Seattle
October 27-30, 2008
The present experimental values:

\[ a_e = 1159652180.73 \times 10^{-12} \]

0.24 parts per billion !! Hanneke et al., PRL100 (2008) 120801

\[ a_\mu = 116592080 (63) \times 10^{-11} \]

0.5 parts per million !! E821 - Final Report: PRD73 (2006) 072003

\[ a_\tau = -0.018 (17) \]

DELPHI - EPJC35 (2004) 159 \([a_\tau^{SM} = 117721(5) \times 10^{-8}, Eidelman \\ & MP '07]\]
The muon g-2: experimental result

Today: $a_\mu^{\text{EXP}} = (116592080 \pm 54_{\text{stat}} \pm 33_{\text{sys}}) \times 10^{-11}$ [0.5ppm].

Future: a new $(g-2)_\mu$ exp? See D. Hertzog’s talk.

Are theorists ready for this? [not yet]
The anomalous magnetic moment: the basics

- The Dirac theory predicts for a lepton $l=e, \mu, \tau$:
  \[ \tilde{\mu}_l = g_l \left( \frac{e}{2m_l c} \right) \vec{s} \quad g_l = 2 \]

- QFT predicts deviations from the Dirac value:
  \[ g_l = 2 \left( 1 + a_l \right) \]

- Study the photon-lepton vertex:
  \[ \bar{u}(p') \Gamma_{\mu} u(p) = \bar{u}(p') \left[ \gamma_{\mu} F_1(q^2) + \frac{i \sigma_{\mu \nu} q^\nu}{2m} F_2(q^2) + \ldots \right] u(p) \]

\[ F_1(0) = 1 \quad F_2(0) = a_l \]
The QED contribution to $a_\mu$

$$a_\mu^{\text{QED}} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right)$$

Schwinger 1948

$$+ 0.765857410 \ (27) \ \left( \frac{\alpha}{\pi} \right)^2$$

Sommerfield; Petermann; Suura & Wichmann '57; Elend '66; MP '04

$$+ 24.05050964 \ (43) \ \left( \frac{\alpha}{\pi} \right)^3$$

Remiddi, Laporta, Barbieri …; Czarnecki, Skrzypek; MP '04;
Friot, Greynat & de Rafael '05

$$+ 130.805 \ (8) \ \left( \frac{\alpha}{\pi} \right)^4$$

Kinoshita & Lindquist '81, …, Kinoshita & Nio '04, '05;
Aoyama, Hayakawa, Kinoshita & Nio, June & Dec 2007

$$+ 663 \ (20) \ \left( \frac{\alpha}{\pi} \right)^5$$

In progress

Kinoshita et al. '90, Yelkhovsky, Milstein, Starshenko, Laporta,
Karshenboim, …, Kataev, Kinoshita & Nio March '06.

Adding up, I get:

$$a_\mu^{\text{QED}} = 116584718.09 \ (14)(04) \times 10^{-11}$$

mainly from 5-loop unc → from new $\delta \alpha$('08)

with $\alpha = 1/137.035999084(51)$ [0.37 ppb]
\[ a_e^{SM} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right) - 0.32847844400290(60) \left( \frac{\alpha}{\pi} \right)^2 \]

\[ A_2^{(4)} \left( \frac{m_e}{m_\mu} \right) = 5.19738670(28) \times 10^{-7} \]
\[ A_2^{(4)} \left( \frac{m_e}{m_\tau} \right) = 1.83762(60) \times 10^{-9} \]
\[ + 1.181234016827(19) \left( \frac{\alpha}{\pi} \right)^3 \]

\[ A_2^{(6)} \left( \frac{m_e}{m_\mu} \right) = -7.37394164(29) \times 10^{-6} \]
\[ A_2^{(6)} \left( \frac{m_e}{m_\tau} \right) = -6.5819(19) \times 10^{-8} \]
\[ A_3^{(6)} \left( \frac{m_e}{m_\mu}, \frac{m_e}{m_\tau} \right) = 1.90945(62) \times 10^{-13} \]
\[ - 1.9144(35) \left( \frac{\alpha}{\pi} \right)^4 \]

\[ + 0.0(4.6) \left( \frac{\alpha}{\pi} \right)^5 \text{ In progress (12672 mass ind. diagrams!)} \]

\[ + 1.682(20) \times 10^{-12} \text{ Hadronic} \]

\[ + 0.0297(5) \times 10^{-12} \text{ Electroweak} \]
The new measurement of the electron g-2 is:
\[ a_e^{\text{exp}} = 1159652180.73 \pm 28 \times 10^{-12} \text{ (Hanneke et al, PRL100 (2008) 120801)} \]

vs. old (factor of 15 improvement, 1.8σ difference):
\[ a_e^{\text{exp}} = 1159652188.3 \pm 4.2 \times 10^{-12} \text{ (Van Dyck et al, PRL59 (1987) 26)} \]

Equating \( a_e^{\text{SM}}(\alpha) = a_e^{\text{exp}} \) → best determination of alpha to date:
\[
\bar{\alpha}^{-1} = 137.035 \, 999 \, 084 \, (12)(37)(2)(33) \, [0.37 \text{ ppb}] \quad \text{Hanneke et al, '08}
\]

\[ \delta a_e^{\text{had}} \quad \delta a_e^{\text{exp}} \quad \text{(smaller than th!)} \]

Compare it with other determinations (independent of \( a_e \)):
\[
\begin{align*}
\bar{\alpha}^{-1} & = 137.036 \, 000 \, 000 \, (110) \, [8.0 \, \text{ ppb}] \quad \text{PRA73 (2006) 032504 (Cs)} \\
\bar{\alpha}^{-1} & = 137.035 \, 998 \, 78 \, (91) \, [6.7 \, \text{ ppb}] \quad \text{PRL96 (2006) 033001 (Rb)}
\end{align*}
\]

\[ \Delta = +0.8 \text{ and } -0.3 \sigma \rightarrow \text{beautiful test of QED at 4-loop level!} \]
Old and new determinations of alpha

Hanneke, Fogwell & Gabrielse, PRL100 (2008) 120801
The Electroweak contribution

- **One-loop term:**

\[
a_\mu^{\text{EW}}(1\text{-loop}) = \frac{5G_\mu m_\mu^2}{24\sqrt{2}\pi^2} \left[ 1 + \frac{1}{5} \left( 1 - 4\sin^2 \theta_W \right)^2 + O \left( \frac{m_\mu^2}{M_{Z,W,H}^2} \right) \right] \approx 195 \times 10^{-11}
\]

1972: Jackiw, Weinberg; Bars, Yoshimura; Altarelli, Cabibbo, Maiani; Bardeen, Gastmans, Lautrup; Fujikawa, Lee, Sanda.

- **One-loop plus higher-order terms:**

\[
a_\mu^{\text{EW}} = 154 (2) (1) \times 10^{-11}
\]

Higgs mass, \( M_{\text{top}} \) error, 3-loop nonleading logs

Hadronic loop uncertainties:

Kukhto et al. ‘92; Czarnecki, Krause, Marciano ‘95; Knecht, Peris, Perrottet, de Rafael ‘02; Czarnecki, Marciano, Vainshtein ‘02; Degrassi, Giudice ‘98; Heinemeyer, Stockinger, Weiglein ‘04; Gribouk, Czarnecki ‘05; Vainshtein ‘03.
The hadronic leading-order (HLO) contribution

\[ \alpha_{\mu}^{\text{HLO}} = \frac{1}{4\pi^2} \int_{4m_{\pi}^2}^{\infty} ds \frac{K(s)\sigma^{(0)}(s)}{s} = \frac{\alpha^2}{3\pi^2} \int_{4m_{\pi}^2}^{\infty} \frac{ds}{s} K(s) R(s) \]

Bouchiat & Michel 1961; Gourdin & de Rafael 1969

F. Jegerlehner, PhiPsi 08, Frascati, April 2008

Central values

Errors \^ 2

Hagiwara et al., PRD 69 (2004) 093003
The HLO contribution: $e^+e^-$ data

Radiative Corrections (Luminosity, ISR, Vacuum Polarization, FSR) are a very delicate issue! Are they all under control?

CMD2’s 1998 $\pi^+\pi^-$ data in the $\rho$ energy range, published in 2007, agree well with their earlier 1995 ones.

SND’s $\pi^+\pi^-$ 2006 data reanalysis appears to be in good agreement with CMD2.

\[ a_\mu^{HLO} = 6909 (39)_{\text{exp}} (19)_{\text{rad}} (7)_{\text{qcd}} \times 10^{-11} \]
\[ = 6894 (42)_{\text{exp}} (18)_{\text{rad}} \times 10^{-11} \quad \text{Hagiwara, Martin, Nomura, Teubner, PLB649(2007)173} \]
\[ = 6923 (60)_{\text{tot}} \times 10^{-11} \quad \text{F. Jegerlehner, PhiPsi 08, Frascati, April 2008} \]
\[ = 6944 (48)_{\text{exp}} (10)_{\text{rad}} \times 10^{-11} \quad \text{de Troconiz & Yndurain, PRD71 (2005) 73008} \]
The RADIATIVE RETURN (ISR) Method: KLOE & BaBar.
Collider operates at fixed energy but $s_{\pi}$ can vary continuously.
Important independent method made possible by beautiful interplay between theory and experiment.


Agreement between KLOE (2008) and CMD2-SND below the $\rho$, some discrepancies above. Their contributions to $a_\mu^{\text{HLO}}$ agree.

News from BaBar. $\pi^+\pi^-$ preliminary results (from 0.5 to 3 GeV) presented at Tau08. Disagreement with CMD2, SND and KLOE. Better agreement with $\tau$ results, especially with Belle.
CMD2 & SND vs KLOE

band: KLOE error
data points: CMD2/SND experiments
CMD-2, SND & KLOE vs BABAR

deviation from 1 of ratio w.r.t. BaBar
stat + syst errors included
The τ data of ALEPH and CLEO are significantly higher than the CMD2-SND-KLOE ones, particularly above the ρ.

The recent $a_\mu^{\pi\pi} \tau$ result of BELLE agrees with Aleph-Cleo-Opal. Some deviations from Aleph’s spectral functions.

Value:

$$a_\mu^{HLO} = 7110 \times 10^{-11}$$

by Davier, Eidelman, Hoecker, Zhang, EPJC31 (2003) 503. NB: Davier & Eidelman chose not to include τ data in their updates of this article until the discrepancy is understood.

Inconsistencies in $e^+e^-$ or τ data? All possible isospin-breaking (IB) effects taken into account? Further recent IB corrections somewhat reduce the diff. with $e^+e^-$ data. Recent claims that $e^+e^-$ & τ data are consistent after IB effects & vector meson mixings considered (Marciano & Sirlin ’88; Cirigliano, Ecker, Neufeld ’01-’02, Flores-Baez et al. ’06 & ’07, Benayoun et al.’07).
Fujikawa, Hayashii, Eidelman [for the Belle Collab.], arXiv:0805.3773, May '08
The hadronic higher-order (HHO) contributions: VP

**HHO: Vacuum Polarization**

\( O(a^3) \) contributions of diagrams containing hadronic vacuum polarization insertions:

\[ a_{\mu}^{\text{HHO}(\text{vp})} = -98 (1) \times 10^{-11} \]

Krause '96, Alemany et al. '98, Hagiwara et al. '03 & '06

Shifts by \(-3 \times 10^{-11}\) if \(\tau\) data are used instead of the \(e^+e^-\) ones

Davier & Marciano '04
The hadronic higher-order (HHO) contributions: LBL

**HHO: Light-by-light contribution**

- Unlike the HLO term, no direct exp. input for the had lbl term. Must rely on theory.
- This term had a troubled life! Its recent determinations vary between:
  - Estimate by Prades, de Rafael, Vainshtein out soon ("g-2 white paper").
  - Erler & Sanchez upper bound: $a_\mu^{HHO}(lbl) < \sim 159 \times 10^{-11}$.
  - Lattice? In progress: Rakow et al (QCDSF), Hayakawa et al.
  - Likely to become the ultimate limitation of the SM prediction.

\[
\begin{align*}
  a_\mu^{HHO}(lbl) &= +80 \ (40) \times 10^{-11} \quad \text{Knecht & Nyffeler '02} \\
  a_\mu^{HHO}(lbl) &= +136 \ (25) \times 10^{-11} \quad \text{Melnikov & Vainshtein '03} \\
  a_\mu^{HHO}(lbl) &= +110 \ (40) \times 10^{-11} \quad \text{Bijnens & Prades '07}
\end{align*}
\]
The muon g-2: Standard Model vs. Experiment

Adding up all the above contribution we get the following SM predictions for \( a_\mu \) and comparisons with the measured value:

\[
\begin{array}{ccc}
\hline
a_\mu^{SM} \times 10^{11} & \Delta a_\mu \times 10^{11} & \sigma \\
\hline
[1] & 116591793 (60) & 287 (87) & 3.3 \\
[2] & 116591778 (61) & 302 (88) & 3.4 \\
[3] & 116591807 (72) & 273 (96) & 2.8 \\
[4] & 116591828 (63) & 252 (89) & 2.8 \\
[5] & 116591991 (70) & 89 (95) & 0.9 \\
\hline
\end{array}
\]

with \( a_\mu^{HHO(lbl)} = 110 (40) \times 10^{-11} \). 

\[ \Delta a_\mu = a_\mu^{EXP} - a_\mu^{SM} \]

- The th error is now the same (or even smaller) as the exp. one!
- If BaBar's prelim. results are used instead, \( \Delta a_\mu \) drops to \(~1.7\sigma\)!

[1] Eidelman at ICHEP06 & Davier at TAU06 (update of ref. [5]).
The muon g-2 and the bounds on the Higgs mass

MP, W.J. Marciano & A. Sirlin

arXiv:0804.1142 [PRD78, 013009 (2008)]
How do we explain $\Delta a_\mu$?

- $\Delta a_\mu$ can be explained in many ways: errors in HHO-LBL, QED, EW, HHO-VP, g-2 EXP, HLO; or New Physics.
- Can $\Delta a_\mu$ be due to hypothetical mistakes in the hadronic $\sigma(s)$?
- An upward shift of $\sigma(s)$ also induces an increase of $\Delta \alpha_{\text{had}}^{(5)}(M_Z)$.

Consider:

$$ a^\text{HLO} : \quad a = \int_{4m^2_\pi}^{s_u} ds \, f(s) \, \sigma(s), \quad f(s) = \frac{K(s)}{4\pi^3}, \quad s_u < M_Z^2, $$

$$ \Delta \alpha_{\text{had}}^{(5)} : \quad b = \int_{4m^2_\pi}^{s_u} ds \, g(s) \, \sigma(s), \quad g(s) = \frac{M_Z^2}{(M_Z^2 - s)(4\alpha\pi^2)}, $$

and the increase

$$ \Delta \sigma(s) = \epsilon \sigma(s) \quad (\epsilon > 0), \quad \text{in the range:} \quad \sqrt{s} \in [\sqrt{s_0} - \delta/2, \sqrt{s_0} + \delta/2] $$
If this shift $\Delta\sigma(s)$ in $[\sqrt{s_0} - \delta/2, \sqrt{s_0} + \delta/2]$ is adjusted to bridge the $g$-2 discrepancy, the value of $\Delta\alpha_{\text{had}}^{(5)}(M_Z)$ increases by:

$$\Delta b(\sqrt{s_0}, \delta) = \Delta a_\mu \frac{\int_{\sqrt{s_0} - \delta/2}^{\sqrt{s_0} + \delta/2} g(t^2) \sigma(t^2) \, t \, dt}{\int_{\sqrt{s_0} - \delta/2}^{\sqrt{s_0} + \delta/2} f(t^2) \sigma(t^2) \, t \, dt}$$

Adding this shift to $\Delta\alpha_{\text{had}}^{(5)}(M_Z) = 0.02768(22)$ [HMNT07], with $\Delta a_\mu = 302(88) \times 10^{-11}$ [HMNT07], we obtain:
EW Bounds on the SM Higgs mass

- The dependence of SM predictions on the Higgs mass, via loops, provides a powerful tool to set bounds on its value.

- Comparing the theoretical predictions of $M_W$ and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ with

  \begin{align*}
  M_W &= 80.399 (25) \text{ GeV} \quad [\text{LEP+Tevatron}] \\
  \sin^2 \theta_{\text{eff}}^{\text{lept}} &= 0.23153 (16) \quad [\text{LEP+SLC}]
  \end{align*}

and

  \begin{align*}
  \Delta \alpha_{\text{had}}^{(5)}(M_Z) &= 0.02768 (22) \quad [\text{HMNT '07}] \\
  M_{\text{top}} &= 172.4 (1.2) \text{ GeV} \quad [\text{CDF-D0, Aug '08}] \\
  \alpha_s(M_Z) &= 0.118 (2) \quad [\text{PDG '08}]
  \end{align*}

we get

$$M_H = 88^{+32}_{-24} \text{ GeV} \quad \& \quad M_H < 145 \text{ GeV} \quad 95\% \text{CL}$$

The value of $\Delta \alpha_{\text{had}}^{(5)}(M_Z)$ is a key input of these EW fits...
The muon g-2: connection with the SM Higgs mass

How much does the $M_H$ upper bound change when we shift $\sigma(s)$ by $\Delta \sigma(s)$ [and thus $\Delta \alpha_{\text{had}}^{(5)}(M_Z)$ by $\Delta b$] to accommodate $\Delta a_\mu$?
The muon g-2: connection with the SM Higgs mass (2)

- The LEP direct-search lower bound is $M_H^{LB} = 114.4$ GeV (95%CL).

- The hypothetical shifts $\Delta \sigma = \varepsilon \sigma(s)$ that bridge the muon g-2 discrepancy conflict with the LEP lower limit when $s_0 > \sim 1.2$ GeV (for bin widths $\delta$ up to several hundreds of MeV).

- While using tau data in the calculation of $a_\mu^{HLO}$ almost solves the muon g-2 discrepancy, it increases the value of $\Delta a_{\text{had}}^{(5)}(M_Z)$, leading to $M_H < 133$ GeV (95%CL), in near conflict with $M_H^{LB}$.

- Recent claim: $e^+e^-$ & tau data consistent below $\sim 1$ GeV (after isospin viol. effects & vector meson mixings). We could thus assume that $\Delta a_\mu$ is fixed by hypothetical errors above $\sim 1$ GeV (where disagreement persists). If so, $M_H^{UB}$ falls below $M_H^{LB}$ !!

- Scenarios where $\Delta a_\mu$ is accommodated without affecting $M_H^{UB}$ are possible, but considerably more unlikely.
How realistic are these shifts $\Delta \sigma(s)$? When compared with the quoted exp. uncertainties? Study the ratio $\varepsilon = \Delta \sigma(s)/\sigma(s)$:

$M_H$ 95% CL u.b. (GeV)
How realistic are these shifts $\Delta \sigma(s)$? (2)

- The minimum $\varepsilon$ is $\sim +4\%$. It occurs if $\sigma$ is multiplied by $(1+\varepsilon)$ in the whole integration region (!), leading to $M_{H}^{UB} \sim 70$ GeV (!!).

- As the quoted exp. uncertainty of $\sigma(s)$ below 1 GeV is $\sim$ a few per cent (or less), the possibility to explain the muon $g-2$ with these shifts $\Delta \sigma(s)$ appears to be unlikely.

- If, however, we allow variations of $\sigma(s)$ up to $\sim 6\%$ ($7\%$), $M_{H}^{UB}$ is reduced to less than $\sim 130$ GeV (131 GeV). E.g., the $\sim 6\%$ shift in the interval $[0.6, 1.2]$ GeV, required to fix $\Delta a_{\mu}$, lowers $M_{H}^{UB}$ to $126$ GeV. Tension with the $M_{H} > 120$GeV “vacuum stability” bound.

Reminder: the above $M_{H}$ upper bounds, like the LEP-EWWG ones, depend on the value of $\sin^2 \theta_{\text{lept}}^{\text{eff}}$. They also depend on $M_{t}$ & its unc. $\delta M_{t}$. We prepared simple formulae to translate easily $M_{H}$ upper bounds discussed above into new values corresponding to $M_{t}$ & $\delta M_{t}$ inputs different from those employed here.
Conclusions

- Beautiful examples of interplay between theory and experiment: \( g_e \) probed at ppt → \( \alpha \) and extraordinary test of QED’s validity; \( g_\mu \) probed at ppt → test of the full SM and great opportunity to unveil (or just constrain) “New Physics” effects!

- The discrepancy \( \Delta a_\mu \) is more than 3 \( \sigma \) if e\(^+\)e\(^-\) data are used. With tau data, the deviation is only ~ 1 \( \sigma \). BaBar 2\( \pi \)? If confirmed, e\(^+\)e\(^-\) data in turmoil! QED & EW solid and ready for exp “Finale”. LBL??

- \( \Delta a_\mu \) can be due to New Physics, or to problems in \( a_\mu^{SM} \) (or \( a_\mu^{EXP} \)). Can it be due to hypothetical mistakes in the hadronic \( \sigma(s) \)? An increase \( \Delta \sigma(s) \) could bridge \( \Delta a_\mu \), leading however to a decrease on the EW upper bound on the SM Higgs mass \( M_H \)...

- By means of a detailed analysis we conclude that solving \( \Delta a_\mu \) via an increase of \( \sigma(s) \) is unlikely in view of current exp. error estimates. However, if this turns out to be the solution, then the \( M_H \) upper bound drops to about 130 GeV which, in conjunction with the LEP 114 GeV direct lower limit, leaves a rather narrow window for \( M_H \).
The End
ALEPH, CLEO & BELLE vs BaBar

BaBar data averaged in wider $\tau$ bins and corrected for $\rho$-$\omega$ interference

M. Davier, Tau08, Novosibirsk, September 2008