The interplay of flavour- and Polyakov-loop-degrees of freedom
A PNJL model analysis

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The QCD Critical Point
Institute for Nuclear Theory, Seattle
Connections between colour and flavour ($N_f = 2$ thermodynamics)

- Flavour blind dofs couple to up- and down- quark densities
- Up- and down- quark densities couple to flavour blind dofs
- Up- and down- quarks *communicate* via an intermediary: Polyakov loop dofs

Quantitative investigation of induced flavour mixing:

- NJL-model + Polyakov-loop model $\rightarrow$ PNJL model
- A perturbative approach\(^1\) to investigate:
  - The Polyakov loop $\langle \Phi \rangle$ and its conjugate $\langle \Phi^* \rangle$
  - Non-vanishing up- and down-quark susceptibilities $\chi_{uu}$ and $\chi_{ud}$

Conclusion & Outlook

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\(^1\)R., Hell, Ratti, Weise arXiv:0712.3152 [hep-ph], [RHRW07]
### Symmetry breaking patterns of QCD at finite $T$

<table>
<thead>
<tr>
<th>Chiral symmetry</th>
<th>Confinement-deconfinement</th>
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<td>1. Explicit breaking $m_q &gt; 0$</td>
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<td>2. Dynamic breaking at low $T$</td>
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<td>3. Order parameter: Chiral condensate $\langle \bar{q}q \rangle$</td>
<td>3. Order parameter: Polyakov loop $\langle \Phi^* \rangle$ and $\langle \Phi \rangle$</td>
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<tr>
<td>4. Quarks are <em>coloured</em> objects</td>
<td>4. Colour <em>screening</em> by vacuum fluctuations of quarks</td>
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</table>

- Dynamic quark masses $\leftrightarrow$ Colour confinement

- Chiral symmetry breaking $\leftrightarrow$ Z(3) symmetry breaking are closely linked

- Joint crossover transition
Modelling colour and flavour dynamics
The Polyakov loop extended Nambu and Jona-Lasinio model (PNJL model)

\[ \text{NJL-model} + \text{Polyakov loop model} = \text{PNJL model} \]

Joint crossover of \( \langle \Phi \rangle \) and \( \langle \bar{q}q \rangle \)

\[ \frac{\langle \bar{\psi}\psi \rangle}{\langle \bar{\psi}\psi \rangle_0} \]

\( T / T_c \)


\[ \langle \Phi \rangle \]

**PNJL:** Joint effects of quarks and Polyakov loop
Confinement (colour) affecting quark densities

S. Rößner, N. Bratović, T. Hell, W. Weise

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Modelling colour and flavour dynamics

The Polyakov loop extended Nambu and Jona-Lasinio model (PNJL model)

NJL-model + Polyakov loop model = PNJL model

Joint crossover of $\langle \Phi \rangle$ and $\langle \bar{q}q \rangle$

Quark densities

$\mu = 0.55 T_c$

PNJL: Joint effects of quarks and Polyakov loop

Confinement (colour) affecting quark densities
Diagrammatic view to quark number densities

\[ n_{q_x} = \frac{\partial \Omega}{\partial \mu_x} = \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \frac{\partial S^{-1}}{\partial \mu_x} S \right] = \times \]

\[ \Omega : \text{thermodynamic potential} \quad \Omega = \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3} \text{tr} \ln \left[ \beta S^{-1} \right] \]

\[ \times : \gamma_0 \tau_x, \text{where } \tau_x \text{ is a matrix in flavour space} \]

Quark number susceptibilities

\[ \chi_{ux} \propto \langle n_u n_x \rangle - \langle n_u \rangle \langle n_x \rangle = \times - \times \times \]

\[ \times = \gamma_0 \tau_u \quad \odot = \gamma_0 \tau_x \]

No explicit isospin breaking (thoughout this presentation)
Introduction of a perturbative interaction $\delta U$

$$\Omega = \Omega_{MF} + \delta U(\zeta) \quad (\Omega_{MF} \text{ indep. of } \zeta)$$

- Propagators remain unchanged
  - Use the same quasiparticles as in MF
- Quark density operators induce new Feynman rules:

$$\begin{align*}
-\cdots &= \frac{\partial S^{-1}}{\partial \zeta} \\
-\cdots &= \left[ \frac{\partial^2 \delta U}{\partial \zeta \partial \zeta} \right]^{-1}
\end{align*}$$

**Corrections to the susceptibilities:**

- $\zeta$ couples to $n_{qx}$
  $$\zeta \xrightarrow{\text{charge conjugation}} -\zeta$$
- $\chi \propto \left[ \frac{\partial^2 \delta U}{\partial \zeta \partial \zeta} \right]^{-1} \neq 0$

- $\zeta$-susceptibility
  $$\chi \propto \left[ \frac{\partial^2 \delta U}{\partial \zeta \partial \zeta} \right]^{-1} \sim a \delta \chi_{ud} \leftrightarrow n_{qx}$-susceptibility

\textsuperscript{a} The mean field contribution of $\chi_{ud}$ vanishes due to flavour symmetry.
Polyakov loop degrees of freedom

- Polyakov loop $\Phi(\vec{x})$ is the trace of a time-like Wilson-line
  \[
  \Phi(\vec{x}) = \frac{1}{N_c} \text{tr}_c L(\vec{x}) \quad L(\vec{x}) = P \exp \left\{ i \int_0^\beta d\tau A^a_4(\vec{x}) t_a \right\}
  \]

- Order parameter for de-confinement
  \[\langle \Phi \rangle = 0 \iff \text{confined} \quad \langle \Phi \rangle \neq 0 \iff \text{deconfined}\]

- Define Polyakov loop fields with good charge conjugation parity:
  \[
  \Phi^+ = \frac{1}{2} \langle \Phi^* + \Phi \rangle \quad \Phi^- = \frac{1}{2} \langle \Phi^* - \Phi \rangle
  \]

QCD toy model (Ginzburg-Landau-type)

\[
S_{QCD}^{\text{eff}} = S_{QCD}^{\text{eff}, 0} + \delta U(\Phi^-)
\]

- Treat $S_{QCD}^{\text{eff}, 0}$ in mean field
- Treat $\delta U$ pertubatively

\[
\chi_{ud} = \chi_{ud}^{MF} + \chi_{ud} \propto -\frac{\partial^2 S_{\text{eff}}}{\partial \mu_u \partial \Phi^-} \left[ \frac{\partial^2 \delta U}{\partial \Phi^- \partial \Phi^-} \right]^{-1} \frac{\partial^2 S_{\text{eff}}}{\partial \mu_d \partial \Phi^-} < 0
\]

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The interplay of flavour- and Polyakov-loop- degrees of freedom
Part 1: NJL model

\[ \mathcal{L}_{NJL} = \bar{\psi} (\mathbf{p} - m_0) \psi - g (\bar{\psi} \gamma^\mu \lambda_a \psi) (\bar{\psi} \gamma_\mu \lambda_a \psi) \]

- Free quarks
- Integrated out gluons
- Local color current interaction
- Chiral symmetry

\[ \text{Local SU}(3)_c \xrightarrow{\text{QCD}\rightarrow\text{NJL}} \text{Global SU}(3)_c \xrightarrow{\text{No confinement}} \]

**Spontaneous chiral symmetry breaking**

\[ \Omega_{NJL} = \frac{\sigma^2 + N^2}{2G} - \frac{T}{2} \sum_{\omega_n} \int \frac{d^3 p}{(2\pi)^3} \text{Tr} \log S^{-1}(\omega_n, \mathbf{p}) \]

- Hartree-Fock approximation using Fierz-transformations
- Bosonization in channels with large 4-quark coupling

\[ \sigma = G \langle \bar{\psi} \psi \rangle \quad M = m_0 - \sigma \quad N = -G \langle \bar{\psi} i \gamma_5 \tau_1 \psi \rangle \]

\[ S^{-1} = \begin{pmatrix}
\mathbf{p} - M + \gamma_0 (\mu + \mu_1) & -i \gamma_5 N \\
-i \gamma_5 N & \mathbf{p} - M + \gamma_0 (\mu - \mu_1)
\end{pmatrix} \]

- No explicit isospin breaking terms (Zhang, Liu [ZL07])

**The interplay of flavour- and Polyakov-loop- degrees of freedom**
Part 2: Polyakov loop model

- Model for SU(3)$_c$-gauge theory ➞ Confinement
  ➞ 1$^{\text{st}}$-order ➞ Spontaneous breakdown of $Z(3)$-center sym.

Order parameter for de-confinement – Polyakov loop

- Polyakov loop $\Phi(\vec{x})$ is the trace of a time-like Wilson-line:
  
  \[
  \Phi(\vec{x}) = \frac{1}{N_c} \text{tr}_c L(\vec{x}) \quad L(\vec{x}) = \mathcal{P} \exp \left\{ i \int_0^\beta d\tau A^a_4(\vec{x}) t_a \right\}
  \]

  ➞ $\langle \Phi \rangle = 0 \iff$ confined ➞ $\langle \Phi \rangle \neq 0 \iff$ deconfined

Ginzburg-Landau effective potential $U = U(\Phi, \Phi^*, T)$

- Simplified loop: $\Phi = \frac{1}{N_c} \text{Tr} \exp \left\{ i A^a_4 t_a \right\} \quad a \in \{3, 8\}$
- Integrate out all dofs that do not change order parameters
  
  \[
  \int \mathcal{D}\Phi \int \mathcal{D}\Phi^* \ e^{-U(\Phi, \Phi^*, T)} = \int \mathcal{D}A \ e^{-S_{\text{eff}}(\Phi(A), \Phi^*(A), T)}
  \]
Polyakov loop model adjusted to lattice QCD data

Ansatz for the Polyakov loop potential (K. Fukushima [Fuk04])

\[
U(\Phi, \Phi^*, T) = -\frac{1}{2} b_2(T) \Phi^* \Phi + \frac{1}{4} b_4(T) \log [J(\Phi, \Phi^*)]
\]

\[
J(\Phi, \Phi^*) = 1 - 6\Phi^* \Phi + 4 \left( \Phi^*^3 + \Phi^3 \right) - 3 (\Phi^* \Phi)^2
\]

\[
b_4(T) = b_4 \left( \frac{T_0}{T} \right)^3 \quad b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2
\]

- Temperature dependent coupling strength \( b_2 = b_2(T) \)

G. Boyd et. al. [B96], O. Kaczmarek et. al. [KKPZ02, KZ05]

- \( \Phi^* = \Phi \) at \( N_f = 0 \): \( U(\Phi, \Phi^*, T) \) only fixed in \( \frac{1}{2}(\Phi^* + \Phi) \)

- Stiffness of \( U(\Phi, \Phi^*, T) \) in \( \frac{1}{2}(\Phi^* - \Phi) \) is free to be adjusted
Polyakov loop extended NJL (PNJL)

- Substitute the Matsubara frequencies $\omega_n$ by $\omega_n + A_4$
  - Formal substitution $\mu \rightarrow \mu - iA_4$ after Matsubara summation

$$\Omega_0 = \Omega_{NJL}|_{\mu \rightarrow \mu - iA_4} + U(\Phi, \Phi^*, T)$$

**Defining mean field as 0\text{th} perturbative order**

- Fermion sign problem: $\mu \rightarrow \mu - iA_4$  $\Rightarrow$  $\Omega_0 = \frac{T}{V} S_E \in \mathbb{C}$
- Identification in 0\text{th} order:  $p(T) = -\Omega_{MF}(T) + \Omega_{MF}(T=0)$
- Mean field:  $\Omega_{MF} = \text{Re} \Omega_0$
  - Maximization of $|e^{-S_E/T}|$ ("quenched" mean field)

$$\frac{\partial \text{Re} \Omega_0}{\partial \sigma} = \frac{\partial \text{Re} \Omega_0}{\partial \Delta} = \frac{\partial \text{Re} \Omega_0}{\partial A_4^{(3)}} = \frac{\partial \text{Re} \Omega_0}{\partial A_4^{(8)}} = 0$$

- Constraints:  $\Omega_{MF} \in \mathbb{R}$  $\Rightarrow$  $\Phi_{MF} = \Phi_{MF}^* \ldots$
Polyakov loop extended NJL (PNJL)

Substitute the Matsubara frequencies $\omega_n$ by $\omega_n + A_4$

- Formal substitution $\mu \rightarrow \mu - iA_4$ after Matsubara summation

$$\Omega_0 = \Omega_{NJL}\big|_{\mu \rightarrow \mu - iA_4} + U(\Phi, \Phi^*, T)$$

Joint crossover of $\langle \Phi \rangle$ and $\langle \bar{q}q \rangle$

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<th>$T_c$ in MeV</th>
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<tr>
<td>Kaczmarek et al. [KZ05] ($N_f = 2$)</td>
</tr>
<tr>
<td>Cheng et al. [C+06] ($N_f = 2+1$)</td>
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<td>PNJL ($N_f = 2$)</td>
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Polyakov loop extended NJL (PNJL)

- Substitute the Matsubara frequencies \( \omega_n \) by \( \omega_n + A_4 \)
  - Formal substitution \( \mu \rightarrow \mu - iA_4 \) after Matsubara summation
  \[
  \Omega_0 = \Omega_{\text{NJL}}|_{\mu \rightarrow \mu - iA_4} + U(\Phi, \Phi^*, T)
  \]

Joint crossover of \( \langle \Phi \rangle \) and \( \langle \bar{q}q \rangle \)

"PNJL-Confinement"

- Confinement at \( T < T_c \):
  - Polyakov loop \( \langle \Phi \rangle \ll 1 \)
  - Free quarks suppressed
- Statistical confinement:
  - Active (color-neutral) quasi-particles:
    \[
    m = 3M \approx M_N
    \]


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The interplay of flavour- and Polyakov-loop- degrees of freedom
Corrections to the PNJL mean field solutions
“Unquenching” the PNJL model

**Goal:** Release constraints on the mean fields

- Integrate out the *approximate* order parameters (mean fields)
  - Subtile cancellation of imaginary parts is guaranteed

**How?** Perturbative expansion about the (constraint) MF solution

- Taylor expansion of the action with respect to the fields

\[
S = \frac{V}{T} \Omega_0 = \frac{V}{T} \sum_k \frac{1}{k!} \omega_k \xi^k \quad \text{with} \quad \xi = \tilde{\theta} - \tilde{\theta}_0
\]

- The vector arrow "\(\tilde{\cdot}\)"

- Set of all fields \(\tilde{\theta} = (\sigma, N, A_4^{(3)}, A_4^{(8)})^T\)

- \(\tilde{\theta}_0\) is the new minimum after SSB

- Separate free and perturbative parts:
  - Free part: \(k = 0, 1, 2\)
  - Interactions: \(k \geq 3\)

**Note:** \(\text{Im}[\omega_1] \neq 0\) for the (former) gauge field \(A_4^{(8)}\)
Expectation values of the Polyakov loop $\langle \Phi \rangle$ and $\langle \Phi^* \rangle$

In mean field MF + corrections

- $\langle \Phi \rangle_{\text{MF}} = \langle \Phi^* \rangle_{\text{MF}}$
- No split of $\langle \Phi \rangle$ and $\langle \Phi^* \rangle$
- $\langle \Phi \rangle \in \mathbb{R}$ and $\langle \Phi^* \rangle \in \mathbb{R}$
- $\langle \Phi \rangle \neq \langle \Phi^* \rangle$ at $\mu \neq 0$

Fluctuation effects beyond mean field produce $\langle \Phi \rangle \neq \langle \Phi^* \rangle$

Susceptibilities: $c_{2u}^{uu}$, $c_{2u}^{ud}$, $c_{4u}^{uu}$, $c_{4u}^{ud}$ beyond mean field

\[ c_n(T) = - \frac{1}{n!} \frac{\partial^n (\Omega / T^4)}{\partial (\mu / T)^n} \bigg|_{\mu = \mu_1 = 0} \]

\[ c_{n}^{l}(T) = - \frac{1}{n!} \frac{\partial^n (\Omega / T^4)}{\partial (\mu_1 / T)^2 \partial (\mu / T)^{(n-2)}} \bigg|_{\mu = \mu_1 = 0} \]

\[ c_{n}^{uu} = \frac{1}{4} (c_n + c_n^{l}) \]

\[ c_{n}^{ud} = \frac{1}{4} (c_n - c_n^{l}) \]

- Isovector moments in agreement with lattice data as well

Lattice data: Allton et al. [A⁺05], Bielefeld-Swansea coll.
Susceptibilities: $c^{uu}_4$, $c^{ud}_2$, $c^{uu}_4$, $c^{ud}_4$ beyond mean field

\[ c_n(T) = -\frac{1}{n!} \frac{\partial^n (\Omega / T^4)}{\partial (\mu / T)^n} \bigg|_{\mu = \mu_1 = 0} \]

\[ c^l_n(T) = -\frac{1}{n!} \frac{\partial^n (\Omega / T^4)}{\partial (\mu / T)^2 \partial (\mu / T)^{(n-2)}} \bigg|_{\mu = \mu_1 = 0} \]

\[ c^{uu}_n = \frac{1}{4} (c_n + c^l_n) \]

\[ c^{ud}_n = \frac{1}{4} (c_n - c^l_n) \]

- Isovector moments in agreement with lattice data as well

Lattice data: Allton et al. [A⁺05], Bielefeld-Swansea coll.
Susceptibilities: $c_{2}^{uu}$, $c_{2}^{ud}$, $c_{4}^{uu}$, $c_{4}^{ud}$ beyond mean field

$$c_n(T) = - \frac{1}{n!} \frac{\partial^n (\Omega/T^4)}{\partial (\mu/T)^n} \bigg|_{\mu=\mu_1=0}$$

$$c_n^{uu} = \frac{1}{4} (c_n + c_n^l)$$

$$c_n^{ud} = \frac{1}{4} (c_n - c_n^l)$$

Isovector moments in agreement with lattice data as well

Lattice data: Allton et al. [A+05], Bielefeld-Swansea coll.

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Susceptibilities: $c_{22}^{uu}, c_{22}^{ud}, c_{44}^{uu}, c_{44}^{ud}$ beyond mean field

\[ c_n(T) = -\frac{1}{n!} \frac{\partial^n(\Omega/T^4)}{\partial(\mu/T)^n} \bigg|_{\mu=\mu_1=0} \]

\[ c_n^{uu} = \frac{1}{4} (c_n + c_n^I) \]

\[ c_n^{ud} = \frac{1}{4} (c_n - c_n^I) \]

Isovector moments in agreement with lattice data as well

Lattice data: Allton et al. [A⁺05], Bielefeld-Swansea coll.

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The interplay of flavour- and Polyakov-loop- degrees of freedom
Susceptibilities: $c_{uu}^2, c_{ud}^2, c_{uu}^4, c_{ud}^4$ beyond mean field

\[ c_n(T) = -\frac{1}{n!} \frac{\partial^n(\Omega/T^4)}{\partial(\mu/T)^n} \bigg|_{\mu=\mu_1=0} \]

\[ c_n^{\mu\mu} = \frac{1}{4} (c_n + c_n^l) \]

\[ c_n^{\mu\text{ud}} = \frac{1}{4} (c_n - c_n^l) \]

Isovector moments in agreement with lattice data as well

Lattice data: Allton et al. [A^+05], Bielefeld-Swansea coll.

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The interplay of flavour- and Polyakov-loop- degrees of freedom
Polyakov loop effective potential adjusted at $N_f = 0$

$$\frac{\delta U(\Phi, \Phi^*, T)}{T^4} \propto \Phi^* \Phi = \frac{\Phi^2 - \Phi^{-2}}{4} \rightarrow \frac{\Phi^2 - k \Phi^{-2}}{4}$$

Thermodynamics unchanged by varying $k$

$c_{2}^{ud}$ with modified $\Phi^-$-potential

$\Phi^-$ at $\mu > 0$

$\Phi^-$-dof $\Rightarrow c_{2}^{ud} < 0$
$c_{2}^{ud}$ controlled by $\Phi^{-} = \frac{1}{2} \langle \Phi^{*} - \Phi \rangle$

$\Phi^{-}$ around $\mu = 0$

$c_{2}^{ud}$ normalized to $\Phi^{-}$-variation

Variations of $\Phi^{-}$ are responsible for $c_{2}^{ud} < 0$. 

Lattice: Döring [PhD Thesis]

Universal for varying $k$
Susceptibilities $\chi_{ud}$ beyond mean field

\[ \frac{\chi_{ux}(T, \mu)}{T^2} = 2c_{2}^{ux} + 12c_{4}^{ux} \left( \frac{\mu}{T} \right)^2 + 30c_{6}^{ux} \left( \frac{\mu}{T} \right)^4 + \cdots \] \quad \text{with } x \in \{ u, d \}

- Fluctuation effects: $\chi_{ud} \neq 0$

Lattice data: Allton et al. [A⁺05], Bielefeld-Swansea coll.
Susceptibilities $\chi_{ud}$ beyond mean field

$$\chi_{ud}/\chi_{uu} \text{ beyond MF}$$

$$\chi_{ux}(T, \mu) = 2c_2^{ux} + 12c_4^{ux} \left( \frac{\mu}{T} \right)^2 + 30c_6^{ux} \left( \frac{\mu}{T} \right)^4 + \cdots \quad \text{with } x \in \{u, d\}$$

- Fluctuation effects: $\chi_{ud} \neq 0$

Lattice data: Allton et al. [A+05], Bielefeld-Swansea coll.
Conclusion

PN JL:
- Chiral symmetry breaking
- Confinement
- Entanglement of chiral and deconfinement crossover

Perturbative approach used to investigate
- Polyakov loop: $\langle \Phi \rangle \neq \langle \Phi^* \rangle$ at $\mu \neq 0$
- Isovector susceptibilities

Variations of $\Phi^-$ are responsible for $c_{2ud} < 0$
- Stiffness of Polyakov loop effective potential directly governs $c_{2ud}$
- $c_{2ud}$ contains information about Polyakov loop effective potential

Outlook
- $2 + 1$ flavors
- Extract Polyakov loop effective potential using $c_{2ud}$ from the lattice
Thank you for your attention

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