Charge Fluctuations and transport coefficients near CEP

Use an effective chiral models and scaling theory to study:

- Charge fluctuations in the presence of spinodal instabilities and their scaling
- Shear, and Bulk viscosities and its scaling near CEP

Krzysztof Redlich, Workshop on QCD-CP, INT
Scaling properties: \( \chi_q = a + b \left| T - T_{TCP} \right|^\theta \)

The strength of the singularity at TCI depends on direction in \((T, \mu_B)\) plane

\( \chi_q \propto \left| T - T_{TCP} \right|^{-1} \) along 1st order line
\( \chi_q \propto \left| T - T_{TCP} \right|^{-1/2} \) any direction not parallel
\( \chi_q \propto \frac{1}{2} \left| T - T_{TCP} \right|^{-1} \) along 2nd order line

Going beyond the mean field:
B.-J. Schaefer & J. Wambach
\( \chi_q \propto \left| T - T_{TCP} \right|^{-0.53(m \neq 0 \Rightarrow 0.78)} \)
FRG: Stokic, Friman & K.R
\( \alpha = (0MF) (-0.25Z(2))(-0.3) \)

C.Sasaki, B. Friman & K.R.
Scaling properties: \( \chi_q = a + b |T - T_{TCP}|^{-\theta} \)

The strength of the singularity at TCP depends on direction in \((T, \mu_B)\) plane.

Going beyond the mean field:

- B.-J. Schaefer & J. Wambach
  \( \chi_q \propto |T - T_{TCP}|^{-0.53(m=0,0.78)} \)

- FRG: Stokic, Friman & K.R
  \( \alpha = (0MF) (-0.25Z(2))(-0.3) \)

\( Q(4) \) univer. class

\( Z(2) \) univer. class

\[
\begin{align*}
\chi_q &\propto |T - T_{TCP}|^{-1} & \text{along 1st order line} \\
\chi_q &\propto |T - T_{TCP}|^{-1/2} & \text{any direction not parallel} \\
\chi_q &\propto \frac{1}{2} |T - T_{TCP}|^{-1} & \text{along 2nd order line}
\end{align*}
\]

See also Y. Hatta, T. Ikeda
Quark and isovector fluctuations along the critical line

NJL-model results: C. Sasaki, B. Friman, K.R.

Non-monotonic behavior of the net quark susceptibility as function of $(T_c, \mu_c)$ in LGT or $\sqrt{s}$ in HIC.

Sensitive probes of TCP/CEP
The nature of the 1\textsuperscript{st} order chiral phase transition

\[ \frac{\partial P}{\partial V} < 0 \quad : \quad \text{stable} \]
\[ \frac{\partial P}{\partial V} > 0 \quad : \quad \text{unstable} \]
\[ \frac{\partial P}{\partial V} = 0 \quad : \quad \text{spinodal} \]

A-B: supercooling (symmetric phase)
B-C: non-equilibrium state
C-D: superheating (broken phase)
Convex anomaly in thermodynamic pressure

NJL model results

\[ m_q = 0 \]

\[ m_q \neq 0 \]

- \( P \) has a cusp at the point where the dynamical quark mass vanishes

- \( P \) is differentiable at all values of \( \mu \)
Dynamical Quark Mass Mass and 

\[ m_q = 0 \]

\[ m_q \neq 0 \]

\( M(\mu, T = \text{fixed}) \)

**Spinodal Lines**: no-unique solution of the gap equation for \( \mu \) between spinodals

**Maxwell constration**: two degenerate minima in the thermodynamic potential
Entropy per baryon and 1\textsuperscript{st} order phase transition

\[ m_q = 0 \quad \text{and} \quad m_q \neq 0 \]

Smooth evolution of entropy/baryon across the 1\textsuperscript{st} order transition
Phase diagram and spinodals

$m_q = 0$

$m_q \neq 0$

mixed phase - Maxwell construction

Critical end point (CEP):

$T = 81\text{ MeV}, \mu = 330\text{ MeV}$

Spinodal lines:

\[
\left( \frac{\partial P}{\partial V} \right)_T = 0 : \text{isothermal}
\]

\[
\left( \frac{\partial P}{\partial V} \right)_S = 0 : \text{isentropic}
\]

\[
\left( \frac{\partial P}{\partial V} \right)_T = \left( \frac{\partial P}{\partial V} \right)_S + \frac{T}{C_V} \left[ \left( \frac{\partial P}{\partial T} \right)_V \right]^2
\]
Quark number susceptibility

- deviation from equilibrium, large fluctuations induced by instabilities

- at 1st order transition point (A, D) : $\chi_q$ is finite
- at isothermal spinodal point (B, C) : $\chi_q$ diverges and changes its sign $\frac{\partial P}{\partial V} < 0$ for stable/meta-stable state $\Rightarrow \frac{\partial P}{\partial V} > 0$ for unstable state
- in unstable region (B-C) : $\chi_q$ is finite and negative
Fluctuations in the chiral limit across spinodals

- Non-singular behavior of fluctuations when crossing spinodal line from the side of symmetric phase:
  \( \Rightarrow \) directly related with a cusp structure of the pressure
Critical exponents at 1\textsuperscript{st} order line and CEP

\[ \chi_q \sim \left| \frac{\mu - \mu_c}{\mu_c} \right|^{-\gamma} \quad \text{with} \quad \gamma_{m_q=0} = \begin{cases} \frac{1}{2} & (0.53) \quad \text{TCP} \\ \frac{1}{2} & 1\text{st} \end{cases}, \quad \gamma_{m_q \neq 0} = \begin{cases} \frac{2}{3} & (0.78) \quad \text{CEP} \\ \frac{1}{2} & 1\text{st} \end{cases} \]

Experimental Evidence for 1\textsuperscript{st} order transition

Specific heat for constant pressure:

\[
C_p = T \left( \frac{\partial S}{\partial T} \right)_p = TV \left[ \chi_{TT} - \frac{2s}{n_q} \chi_{\mu T} + \frac{s^2}{n_q^2} \chi_q \right]
\]


negative heat capacity: anomalously large fluctuations
\[\Rightarrow\] an evidence of the 1\textsuperscript{st} order liquid-gas phase transition

Low energy nuclear collisions
Net-quark fluctuations on spinodals

at any spinodal points:

\[-V \frac{\partial P}{\partial V} \bigg|_T = \frac{n_q^2}{\chi_q} = \frac{1}{\text{compres.}}\]

Singularity at CEP are the remnant of that along the spinodals

Transport coefficients near CEP

- Minimum of shear viscosity may indicate the location of $T_c$ : L. Csernai, J. Kapusta & L. McLerran (06)
- Divergent bulk viscosity at CEP from QCD trace anomaly argument: D. Kharzeev & Tuchin 07
  
  F. Karsch, D. Kharzeev & Tuchin 08

Our goal: C. Sasaki & K.R. hep-ph 0806.4745

- Find viscosities in quasi particle models with dynamically generated mass $M(T, \mu) = m_{bare} + f(T, \mu)$, under relaxation time approximation
- Derive scaling behavior of viscosities in O(4) and Z(2) universality class using scaling theory
Transport Coefficient near phase transition

L. Csernay, J. Kapusta & L. McLerran 06
R. Lacey et al. 07: data for shear viscosity

Harvey B. Meyer 08 LGT results for bulk viscosity, SU(3) gauge th.

Shear viscosity to entropy ratio in LGT

\[
\frac{\eta}{s} = \begin{cases} 
0.134(33) & \text{for } T = 1.65T_c \\
0.102(56) & \text{for } T = 1.24T_c 
\end{cases}
\]
Transport coefficients from kinetic theory:

- Energy momentum tensor

\[ T^{\mu\nu} = \int \frac{d^3 p}{(2\pi)^3} p^\mu p^\nu \frac{1}{E} \left[ f + \overline{f} \right] \]

\[ E^2 = \overrightarrow{p}^2 + M^2 (T, \mu) \]

- Assume small deviations from equilibrium \( \delta f = f - f_0 \)
  with \( f^{-1} = \exp(E - \overrightarrow{p} \cdot \overrightarrow{u} \mp \mu) \pm 1 \), consequently

\[ \delta T^{\mu\nu} = \int \frac{d^3 p}{(2\pi)^3} p^\mu p^\nu \frac{1}{E} \left[ \delta f + \delta \overline{f} \right] \]
To get $\delta f$ use Boltzmann equation under relaxation time approximation

$$\frac{\partial f}{\partial t} + \vec{v}_p \nabla f = -C[f] \approx -\frac{f - f_0}{\tau}$$

$$\Rightarrow \ p^\mu \partial_\mu f_0 = -\frac{E}{\tau} \delta f$$

with the collision time obtained from the particle density and cross section

$$\tau_f^{-1} = n_f <\sigma \nu_{rel}>$$
derivation of transport coefficients:

use:

- $\delta f$ from Boltzmann equation:  $\delta f = -\tau E^{-1} p^\mu \partial_\mu f_0$
- energy conservation:  $\partial_0 T^{00} = 0$
- charge conservation:  $\partial_0 j^0 = 0$
- stationary condition:  $\partial P / \partial M = 0$
- thermodynamics relation:  $\varepsilon = T \partial P / \partial T - P + \mu \partial P / \partial \mu$

get:

$$\delta T^{ij} = -\zeta \delta_{ij} \partial_k u^k - \eta W_{ij}$$
shear viscosity $\eta(T, \mu)$ is not modified by thermal change of quasi-particle energy

$$\eta = \frac{g}{15T} \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{E^2} [\tau f_0 (1 \pm f_0) + \bar{\tau} \bar{f}_0 (1 \pm \bar{f}_0)]$$

the above coincides with Hosoya & Kajantie (1985), however, the bulk viscosity
\( \zeta(T, \mu) \) is modified by thermal change of quasi-particle energy through: \( \frac{\partial E}{\partial T} \) and \( \frac{\partial E}{\partial \mu} \)

\[
\zeta = -\frac{g}{3T} \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{M^2}{E} \left( \tau f_0 (1 \pm f_0) + \tau \bar{f}_0 (1 \pm \bar{f}_0) \right) 
\times \left( \frac{\vec{p}^2}{3E} - \left( \frac{\partial P}{\partial \varepsilon} \right)_n \left( E - T \frac{\partial E}{\partial T} - \mu \frac{\partial E}{\partial \mu} \right) + \left( \frac{\partial P}{\partial n} \right)_\varepsilon \frac{\partial E}{\partial \mu} \right) 
- \frac{M^2}{E} \left( \tau f_0 (1 \pm f_0) - \tau \bar{f}_0 (1 \pm \bar{f}_0) \right) \left( \frac{\partial P}{\partial n} \right)_\varepsilon \right] 
\]

- for \( \frac{\partial E}{\partial T}(\partial \mu) = 0 \) the above coincides with Hosoya & Kajantie (1985)
- For \( \frac{\partial E}{\partial \mu} = 0 \) the above coincides with Arnold, Dogan & Moore (06)
Near phase transition the bulk viscosity can be singular through derivatives terms:

\[
\frac{\partial E}{\partial x} = \frac{1}{2E} \frac{\partial M^2}{\partial x} \quad \chi_{xy} = \frac{\partial^2 P}{\partial x \partial y} \quad \text{with} \quad x, y = (T, \mu)
\]

\[
\left. \frac{\partial P}{\partial \varepsilon} \right|_n = \frac{1}{C_V \chi_{\mu \mu}} (s \chi_{\mu \mu} - n \chi_{\mu T})
\]

\[
\left. \frac{\partial P}{\partial n} \right|_\varepsilon = \frac{1}{C_V \chi_{\mu \mu}} (nT \chi_{TT} + (n \mu - sT) \chi_{\mu T} - s \mu \chi_{\mu \mu})
\]

\[
C_V = T \left( \frac{\partial S}{\partial T} \right) \bigg|_V = T \left( \chi_{TT} - \frac{\chi_{\mu T}^2}{\chi_{\mu \mu}} \right)
\]

which all can diverge at the critical points!
critical behavior of bulk viscosity:

- consider a system where dynamical mass acts as an order parameter

- Mean Field Scaling from Ginzburg-Landau potential

\[ \Omega(T, \mu, M) = \Omega_{M=0} + a(T, \mu) M^4 + b(T, \mu) M^2 - hM \]

=> 2nd order: \(a=0\) and \(b>0\)  TCP at \(a=b=0\)

with

\[ a(T, \mu) = \alpha |T - T_c| + \beta |\mu - \mu_c| \]

and from gap equation

\[ \zeta_{\text{singular}} \bigg|_{\text{TCP}} \sim \frac{M^3}{C_V} \left( \frac{\partial M}{\partial T} - \frac{\alpha}{\beta} \frac{\partial M}{\partial \mu} \right) \sim M^4 \chi_\sigma \times 0 \sim t^{2/1-1/2} \times 0 \approx 0 \]

\[ \zeta_{\text{singular}} \bigg|_{2^{\text{nd}}} \sim M^2 = 0 \]

Conclusions: under MF approximation there is no singularity of bulk viscosity at the TCP and 2nd order phase transition.

Actually: in the chiral limit the bulk viscosity vanishes at the transition point.
\[ F_S(T, \mu) = t^{2-\alpha} \int_S(t^{-\beta} \delta h) \quad t \equiv t + A\bar{\mu}^2, \quad t = |T - T_c| / T_c, \quad \bar{\mu} = \mu / T_c \]

that gives \( \chi_{\mu\mu} \sim t^{1-\alpha}, \quad \chi_{TT} \sim t^{-\alpha}, \quad C_V \sim t^{-\alpha}, \quad M \sim t^\beta \)

consequently
\[ \zeta^{\text{singular}} \sim \frac{M^3}{C_V} \frac{\partial M}{\partial T} \sim t^{\alpha+4\beta-1} \]

for \( \alpha = -0.24, \quad \beta = 0.38 \quad \Rightarrow \quad \zeta^{\text{singular}} \sim t^{+0.28} \rightarrow 0 \)

Conclusions: bulk viscosity is non-singular at O(4) critical point
- **Z(2) scaling of bulk viscosity: use the free energy**

\[
F_S(T, \mu) = h^{1+\delta} f_S(h^{-1/\beta\delta} t)
\]

\[
t \equiv A_{t,h} t + B_{t,h} \mu, \quad x = |x - x_c| / x_c
\]

that gives

\[
\chi_{\mu\mu,\mu T,TT} \sim h^{-\gamma/\beta\delta}, \quad C_V \sim h^{-\gamma/\beta\delta}, \quad M \sim h^{1/\delta}
\]

consequently

\[
\zeta^{\text{singular}} \sim \frac{M^3}{C_V} \frac{\partial M}{\partial T} \sim h^{\gamma/\beta\delta+4/\delta-1}
\]

for \(\gamma = 1.25, \beta = 0.31, \delta = 5.2\) \(\Rightarrow\) \(\zeta^{\text{singular}} \sim h^{+0.54} \to 0\)

**Conclusions:** bulk viscosity is non-singular at Z(2) critical point within the relaxation time approximation.
Beyond the relaxation time approximation

- Modes with long wave-length could stay in non-equil.
- $\zeta$ modified by dynamic critical exponents “$z$”

Onuki, “Phase transition dynamics”, Cambridge Univ. Press (02)

For O(4):

$$\zeta_{\text{sin gular}} \sim t^{-z\nu + \alpha}$$

For Z(2):

$$\zeta_{\text{sin gular}} \sim t^{-z^* + \alpha}$$

Singularity in bulk viscosity along O(4) critical line and at the TCP due to dynamic critical exponents
- How relevant $\partial M/\partial T$ is: $\zeta$ vs. $\zeta^{(0)}$ w/o $\partial M/\partial T$
  a demonstration in NJL model under MF approximation at $\mu = 0$

- Large change of bulk viscosity due to contribution from temperature derivative of dynamical quark mass, $\partial M/\partial T$
Bulk Viscosity across the phase transition in the NJL model

- Cross over region
- CEP and 1st order line
Bulk viscosity to entropy ratio

- Cross over line
- CEP and 1\textsuperscript{st} order line
Shear viscosity normalized by momentum cutoff

- Cross over transition
- CEP and 1\textsuperscript{st} order line
Shear viscosity per entropy

- Crossover
- CEP and 1\textsuperscript{st} order line
Bulk and shear viscosity along the phase boundary

- shear viscosity
- bulk viscosity
Conclusions

- A non-monotonic change of the net-quark susceptibility probes the existence of CEP. However in non-equilibrium: due to spinodal instability, the charge fluctuations diverge at 1\textsuperscript{st} order critical line.

  \[ \Rightarrow \text{Large fluctuations signals 1}\textsuperscript{st} \text{ order transition} \]

- Under relaxation time approximation the bulk viscosity is finite at CEP and O(4) line.

  \[ \Rightarrow \text{Divergence of bulk viscosity controlled by the dynamical, rather than static critical exponents} \]