Lattice results on the QCD critical point

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Where is the critical point?
Bulk thermodynamics with non-vanishing chemical potential

\[ Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_E(V, T, \mu)} \]

\[ = \int \mathcal{D}A \left[ \text{det} \ M(\mu) \right]^f e^{-S_G(V, T)} \]

↑ complex fermion determinant;
Bulk thermodynamics with non-vanishing chemical potential

\[
Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_E(V, T, \mu)}
= \int \mathcal{D}A \ [\det M(\mu)]^f \ e^{-S_G(V, T)}
\]

\[\uparrow\text{complex fermion determinant;}\]

ways to circumvent this problem:

- reweighting: works well on small lattices; requires exact evaluation of \[\det M\]

- Taylor expansion around \(\mu = 0\): works well for small \(\mu\);

- imaginary chemical potential: works well for small \(\mu\); requires analytic continuation

- canonical ensemble: need to evaluate fermion determinant
Early lattice results

Reweighting:

\[(6 - 12)^3 \times 4\]

Reweighting and Lee-Yang zeroes

\[ Z_{\text{norm}}(\beta_{\text{Re}}, \beta_{\text{Im}}, \mu) \equiv \frac{Z(\beta_{\text{Re}}, \beta_{\text{Im}}, \mu)}{Z(\beta_{\text{Re}}, 0, 0)} \]

\[ = \left\langle e^{6i\beta_{\text{Im}}N_{\text{site}}\Delta t} e^{i\theta} \left| e^{\left(N_f/4\right)(\ln \det M(\mu) - \ln \det M(0))} \right\rangle(\beta_{\text{Re}}, 0, 0) \right\rangle \]

additional complication in QCD phase of fermion matrix contributes

\[ e^{i\theta} \sim e^{i\left[c\mu V + \mathcal{O}(\mu^3 V)\right]} \]

the sign problem severely limits the range of applicability of the reweighting method

S. Ejiri, PR D 73 (2006) 054502

SU(3):

\[ e^{i\theta} \equiv 1 \]
Status 2008: ???

- the devil's advocates

  imaginary $\mu$: Ph. de Forcrand, O. Philipsen, NP B642 (2002) 290;
  updated at Lattice 08 and xQCD 08

- the convinced

  Taylor expansion: R. Gavai, S. Gupta, PR D71 (2005) 114014;
  arXiv:0806.2233

<table>
<thead>
<tr>
<th>$N_\tau$</th>
<th>$T_c(\mu_c)/T_c(0)$</th>
<th>$\mu_c/T$</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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<td>2005: 2-flavor; standard staggered</td>
</tr>
<tr>
<td>6</td>
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<td>1.8(1)</td>
<td>2008: 2-flavor; no $V \to \infty$</td>
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Phase diagram for $\mu_B = 0$

- already the $\mu_B = 0$ phase diagram is not fully explored
- phase boundary is known to be very sensitive to cut-off effects

$N_\tau = 4$, standard staggered fermions (the setup used by deF+P):

$\Rightarrow m_{ps}^{crit} \approx 300$ MeV for $n_f = 3$, i.e. larger than physical $m_\pi$

FK, E. Laermann, C Schmidt, PL B520 (2001) 41
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(also $N_\tau = 6$, unimp.)
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![Phase diagram](image)

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Mean field argument

- Sigma model prediction near tri-critical point on the ms axis.

\[ V_{\text{eff}}(\sigma) = \frac{1}{2} a\sigma^2 + \frac{1}{4} b\sigma^4 + \frac{1}{6} c\sigma^6 - h\sigma \]

Critical point: \[ \frac{\partial^n V_{\text{eff}}}{\partial \sigma^n} = 0 \quad (n=1,2,3) \]

\[ m_{\text{ud}}^{\text{crit}} \sim (m_T - m_s)^{5/2} \]

\[ m_{\text{ud}}^{\text{crit}} \sim \mu^5 \]

\[ b \sim (m_T - m_s) \quad \Rightarrow \quad b \sim \mu^2 \]
Bulk thermodynamics for small $\mu_q/T$

- Taylor expansion of pressure up to $\mathcal{O}\left((\mu_q/T)^6\right)$

\[
\frac{p}{T^4} = \frac{1}{VT^3} \ln Z = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n \simeq c_0 + c_2 \left(\frac{\mu_q}{T}\right)^2 + c_4 \left(\frac{\mu_q}{T}\right)^4 + c_6 \left(\frac{\mu_q}{T}\right)^6
\]

quark number density
\[
\frac{n_q}{T^3} = 2c_2 \frac{\mu_q}{T} + 4c_4 \left(\frac{\mu_q}{T}\right)^3 + 6c_6 \left(\frac{\mu_q}{T}\right)^5
\]

quark number susceptibility
\[
\frac{\chi_q}{T^2} = 2c_2 + 12c_4 \left(\frac{\mu_q}{T}\right)^2 + 30c_6 \left(\frac{\mu_q}{T}\right)^4
\]

an estimator for the radius of convergence
\[
\left(\frac{\mu_q}{T}\right)_{crit} = \lim_{n \to \infty} \left| \frac{c_{2n}}{c_{2n+2}} \right|^{1/2}
\]

$c_n > 0$ for all $n$

$\Rightarrow$ singularity for real $\mu$
Estimating $T_c(\mu_c)$ and $\mu_c/T$

How can we estimate the location of the critical endpoint from expansion coefficients of the Taylor series?
Estimating $T_c(\mu_c)$ and $\mu_c/T$

How can we estimate the location of the critical endpoint from expansion coefficients of the Taylor series?

The approach taken by Gavai and Gupta: arXiv:0806.2233

..As one comes down in $T$ from large values of $T$, the series coefficients remain positive down to some lowest value of $T/T_c$....At this temperature the radius of convergence is...
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a potential problem with unimproved staggered fermions:

unimproved staggered fermions have $\mu$-dependent $O(\alpha^2)$ corrections that are positive and significant even in the ideal gas limit. For instance, they generate a positive $O(\mu^6)$ term


$$O(\alpha^2) \sim \left( 1 + \frac{147}{31} \left( \frac{\mu}{\pi T} \right)^2 + \frac{105}{31} \left( \frac{\mu}{\pi T} \right)^4 + \frac{21}{31} \left( \frac{\mu}{\pi T} \right)^6 \right) \frac{1}{N_f^2}$$
Bulk thermodynamics for small $\mu_q/T$

- Taylor expansion of pressure up to $O((\mu_q/T)^6)$

$N_T = 4$, $O(a^2)$ improved staggered fermions, $m_\pi \simeq 770$ MeV

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z = \sum_{n=0}^{\infty} c_n(T) \left( \frac{\mu_q}{T} \right)^n \simeq c_0 + c_2 \left( \frac{\mu_q}{T} \right)^2 + c_4 \left( \frac{\mu_q}{T} \right)^4 + c_6 \left( \frac{\mu_q}{T} \right)^6$$

C. Allton et al, PR D71 (2005) 054508
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$c_n > 0$ for all $n$ and $T \lesssim 0.95 \ T_c \iff$ singularity for real $\mu$ (positive $\mu^2$)
Bulk thermodynamics for small $\mu_q/T$

Taylor expansion of pressure up to $\mathcal{O} \left( \left( \frac{\mu_q}{T} \right)^6 \right)$

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irregular sign of $c_n$ for $T \gtrsim T_c$ $\iff$ singularity in complex plane
• Results for expansion coefficients $c_{i,j,k}^{u,d,s}$

**Taylor expansion of the pressure**

- **Cut-off dependence:**
  - Small cut-off effects in the transition region (similar to p, e-3p, ...)

- **Mass dependence:**
  - $T_c$ decreases with decreasing mass
  - Fluctuations increase with decreasing mass

- **Preliminary**

  - $n_f=2+1$, $m_\pi=220$ MeV
  - $n_f=2$, $m_\pi=770$ MeV

  - $c_2^u$
    - Red: RBC-Bielefeld, preliminary

  - $c_4^u$
    - Filled: $n_f=4$
    - Open: $n_f=6$

  - $c_6^u$
    - Filled: $n_f=4$
    - Open: $n_f=6$

- **SB**

- **Preliminary**
Hadronic fluctuations at $\mu_q = 0$

- expect 2$^{nd}$ order transition in 3-d, O(4) symmetry class;

scaling field: $t = \left| \frac{T - T_c}{T_c} \right| + A \left( \frac{\mu_q}{T_c} \right)^2$, $\mu_{crit} = 0$

singular part: $f_s(T, \mu_u, \mu_d) = b^{-1} f_s(tb^{1/(2-\alpha)}) \sim t^{2-\alpha}$

$c_2 \sim \frac{\partial^2 \ln Z}{\partial \mu^2_q} \sim t^{1-\alpha}$, $c_4 \sim \frac{\partial^4 \ln Z}{\partial \mu^4_q} \sim t^{-\alpha}$ ($\mu = 0$)

- O(4)/O(2): $\alpha < 0$, small $\Rightarrow$

$c_2 \sim \langle (\delta N_q)^2 \rangle$ dominated by T-dependence of regular part

$c_4 \sim \langle (\delta N_q)^4 \rangle - 3 \langle (\delta N_q)^2 \rangle^2$ develops a cusp

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$$c_2 \sim \frac{\partial^2 \ln Z}{\partial \mu_q^2} \sim t^{1-\alpha}, \quad c_4 \sim \frac{\partial^4 \ln Z}{\partial \mu_q^4} \sim t^{-\alpha} \quad (\mu = 0)$$

$$\epsilon \sim \frac{\partial \ln Z}{\partial T} \sim t^{1-\alpha}, \quad C_V \sim \frac{\partial^2 \ln Z}{\partial T^2} \sim t^{-\alpha} \quad (\mu = 0)$$

$\Rightarrow$ 2nd derivative w.r.t $\mu_q$ "looks like energy density"

$\Rightarrow$ 4th derivative w.r.t $\mu_q$ "looks like specific heat"
Generic expansion coefficients

similar in PNJL model: S. Roessner et al, PR D75 (2007) 034007
Estimating $T_c(\mu_c)$ and $\mu_c/T$

How can we estimate the location of the critical endpoint from expansion coefficients of the Taylor series?

- need $c_n(T) > 0$ to have a singularity on the real axis
- expect hadron resonance gas to be a good approximation at low $T$:
  \[ c_n^{HRG} > 0 \text{ for all } n, \text{ but } r_n^{HRG} = \sqrt{1/(n + 2)/(n + 1)} \to 0 \]

⇒ conjecture:

the position of the first maximum of $c_n(T)$, e.g. at $T_n < T_c(0)$, gives an upper bound on $T_c(\mu_c)$ as one will find $c_{n+2}(T) < 0$ for $T > T_n$

- This is completely opposite to the criterium formulated by Gavai and Gupta
• **Consequences for the phase diagram:**

The radius of convergence can be estimated from the Taylor coefficients of the pressure:

\[ \rho = \lim_{n \to \infty} \rho_n \]

with

\[ \rho_n = \sqrt{\frac{c_n^B}{c_{n+2}^B}} \]

• for \( T > T_c \), \( \rho_n \to \infty \)

• for \( T < T_c \), \( \rho_n \) is bound by the transition line

\[ \frac{p}{T^4} = G(T) + F(T) \cosh \left( \frac{\mu_B}{T} \right) \]

\[ \to \rho_n = \sqrt{\frac{1}{(n + 2)(n + 1)}} \]

→ look for non-monotonic behavior in the radius of convergence
Estimating $T_c(\mu_c)$ and $\mu_c/T$

Status of the RBC-BI project

- calculations for $N_\tau = 4$ and 6; $N_\sigma = 4N_\tau$
- uses an $\mathcal{O}(a^2)$ improved staggered action (p4fat3)

estimator for $\mu_c$:

$$
\left( \frac{\mu_c(T)}{T_c(0)} \right)_n \equiv \rho_n = \frac{T}{T_c(0)} \sqrt{c_n \frac{c_{n+2}}{c_{n+1}}}
$$

![Graph showing slight quark mass dependence](image)

- slight quark mass dependence
Estimating $T_c(\mu_c)$ and $\mu_c/T$

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- slight quark mass dependence
- weak cut-off dependence
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![Graph showing $T / T_c(0)$ vs. $\mu_B / T_c(0)$]

- slight quark mass dependence
- weak cut-off dependence
- $O(\mu^6)$ requires more statistics
Estimating $T_c(\mu_c)$ and $\mu_c/T$

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Hadronic fluctuations at $\mu > 0$ from Taylor expansion coefficients at $\mu = 0$

$n_f = 2$, $m_\pi \simeq 770$ MeV: S. Ejiri, FK, K.Redlich, PLB633 (2006) 275

$n_f = 2 + 1$, $m_\pi \simeq 220$ MeV: RBC-Bielefeld, preliminary

- Taylor expansion of bulk thermodynamics in terms of $\mu_{u,d,s}$

$$\frac{p}{T^4} \equiv \frac{1}{VT^3} \ln Z(V, T, \mu_u, \mu_d, \mu_s)$$

$$= \sum_{i,j,k} c_{i,j,k} \left( \frac{\mu_u}{T} \right)^i \left( \frac{\mu_u}{T} \right)^j \left( \frac{\mu_s}{T} \right)^k$$

- Expansion coefficients evaluated at $\mu_{u,d,s} = 0$ are related to fluctuations of $B, S, Q$ at $\mu_{B,S,Q} = 0$:

↑ baryon number, strangeness, charge fluctuations

event-by-event fluctuations at RHIC and LHC
Hadronic fluctuations at $\mu > 0$ from Taylor expansion coefficients at $\mu = 0$

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quadratic and quartic fluctuations

$$\chi^x_2 = \frac{\partial^2 p/T^4}{\partial (\mu_x/T)^2} = \frac{1}{VT^3} \langle (\delta N_x)^2 \rangle_{\mu=0} = \frac{1}{VT^3} \langle N_{x}^2 \rangle_{\mu=0}$$

$$\chi^x_4 = \frac{\partial^4 p/T^4}{\partial (\mu_x/T)^4} = \frac{1}{VT^3} \left( \langle (\delta N_x)^4 \rangle - 3\langle (\delta N_x)^2 \rangle^2 \right)_{\mu=0}$$

$$= \frac{1}{VT^3} \left( \langle N_{x}^4 \rangle - 3\langle N_{x}^2 \rangle^2 \right)_{\mu=0}$$

with $x = u, \ d, \ s$ or $B, \ Q, \ S$
Quadratic fluctuations of baryon number charge & strangeness in (2+1)-flavor QCD

vanishing chemical potentials:

\[ \chi^Q_2 = \frac{1}{VT^3} \langle Q^2 \rangle \]
\[ \chi^B_2 = \frac{1}{VT^3} \langle N^2_B \rangle \]
\[ \chi^S_2 = \frac{1}{VT^3} \langle N^2_S \rangle \]

rapid approach to SB limit

\[ \Rightarrow \text{ smooth change of quadratic fluctuations across transition region} \]

chiral limit: \( \chi^B_2, \chi^Q_2 \sim |T - T_c|^{1-\alpha} + \text{regular} \)
vanishing chemical potentials:

\[
\chi_Q^4 = \frac{1}{VT^3} \left( \langle Q^4 \rangle - 3\langle Q^2 \rangle^2 \right)
\]

\[
\chi_B^4 = \frac{1}{VT^3} \left( \langle N_B^4 \rangle - 3\langle N_B^2 \rangle^2 \right)
\]

\[
\chi_S^4 = \frac{1}{VT^3} \left( \langle N_S^4 \rangle - 3\langle N_S^2 \rangle^2 \right)
\]

⇒ large light quark number & charge fluctuations across transition region

\[
\text{chiral limit: } \chi_B^4, \chi_Q^4 \sim |T - T_c|^{-\alpha} + \text{regular}
\]
Ratios of quartic and quadratic fluctuations of charges in (2+1)-flavor QCD

\[ n_f = 2: \text{S. Ejiri, FK, K.Redlich, PLB633 (2006) 275} \]
\[ n_f = 2 + 1: \text{RBC-Bielefeld, preliminary} \]

\[ \frac{\text{charged particle number}}{\text{flavor number}} \]

\[ \text{baryon number fluctuation} \]

\[ \text{charge fluctuation} \]

**Chiral limit:** ratios \( \sim |T - T_c|^{-\alpha} + \text{regular} \)

\[ \Rightarrow \text{enhancement over resonance gas values? (need to improve } N_T = 6) \]

\[ \Rightarrow \text{may be observable in event-by-event fluctuations} \]

**Quark sector quickly** \( (T \gtrsim 1.5T_c) \) behaves perturbative
Quark number in Boltzmann approximation

baryonic sector of pressure in a hadron resonance gas;

\[ m_B \gg T \Rightarrow \text{Boltzmann approximation: } p_B/T^4 = \sum_{m \leq m_{\text{max}}} p_m/T^4 \]

with \[ p_m/T^4 = F(T, m, V) \cosh(B \mu_B/T) \]

\[ \chi_2^B \equiv \frac{\partial^2 p_m/T^4}{\partial (\mu_B/T)^2} = B^2 F(T, m, V) \cosh(B \mu_B/T) \]

\[ \chi_4^B \equiv \frac{\partial^4 p_m/T^4}{\partial (\mu_B/T)^4} = B^4 F(T, m, V) \cosh(B \mu_B/T) \]

ratio of fourth (\( \chi_4^B \)) and second (\( \chi_2^B \)) cumulant of quark number fluctuation gives "unit of charge" carried by the particle with mass "m":

\[ m \gg T \Rightarrow R_{4,2}^B \equiv \frac{\chi_4^B}{\chi_2^B} = B^2 \]
Charge fluctuations in Boltzmann approximation

- **hadronic resonance gas**: contributions from isosinglet \( G^{(1)} : \eta, \ldots \) and isotriplet \( G^{(3)} : \pi, \ldots \) mesons as well as isodoublet \( F^{(2)} : p, n, \ldots \) and isoquartet \( F^{(4)} : \Delta, \ldots \) baryons

\[
\frac{p(T, \mu_q, \mu_I)}{T^4} \simeq G^{(1)}(T) + G^{(3)}(T) \frac{1}{3} \left( 2 \cosh \left( \frac{2\mu_I}{T} \right) + 1 \right) \\
+ F^{(2)}(T) \cosh \left( \frac{3\mu_q}{T} \right) \cosh \left( \frac{\mu_I}{T} \right) \\
+ F^{(4)}(T) \frac{1}{2} \cosh \left( \frac{3\mu_q}{T} \right) \left[ \cosh \left( \frac{\mu_I}{T} \right) + \cosh \left( \frac{3\mu_I}{T} \right) \right]
\]

- **charge fluctuations** at \( \mu_q = \mu_I = 0 \);
  - isospin quartet \( F^{(4)} \) contains baryons carrying charge 2

\[
R_{4,2}^Q \equiv \frac{\chi_4^Q}{\chi_2^Q} = \frac{4G^{(3)} + 3F^{(2)} + 27F^{(4)}}{4G^{(3)} + 3F^{(2)} + 9F^{(4)}} \rightarrow 1 \text{ for } T \rightarrow 0
\]

contribution of doubly charged baryons increases quartic relative to quadratic fluctuations
Hadronic fluctuations at $\mu_q = 0$

- Expect $2^{nd}$ order transition in 3-d, O(4) symmetry class;

scaling field: $t = \left| \frac{T - T_c}{T_c} \right| + A \left( \frac{\mu_q}{T_c} \right)^2, \quad \mu_{\text{crit}} = 0$

singular part: $f_s(T, \mu_u, \mu_d) = b^{-1} f_s(tb^{1/(2-\alpha)}) \sim t^{2-\alpha}$

$$c_2 \sim \frac{\partial^2 \ln Z}{\partial \mu_q^2} \sim t^{1-\alpha}, \quad c_4 \sim \frac{\partial^4 \ln Z}{\partial \mu_q^4} \sim t^{-\alpha} \quad (\mu = 0)$$

- O(4)/O(2): $\alpha < 0$, small $\Rightarrow$

$c_2 \sim \langle (\delta N_q)^2 \rangle$ dominated by T-dependence of regular part

$c_4 \sim \langle (\delta N_q)^4 \rangle - 3 \langle (\delta N_q)^2 \rangle^2$ develops a cusp

Y. Hatta, T. Ikeda, PRD67 (2003) 014028
(Hadronic) Fluctuations at $\mu_q > 0$

- Expect $2^{nd}$ order transition in 3-d, O(4) symmetry class;

scaling field: $t = \left| \frac{T - T_c}{T_c} \right| + A \left( \left( \frac{\mu_q}{T_c} \right)^2 - \left( \frac{\mu_{crit}}{T_c} \right)^2 \right)$

\[ \uparrow \ A \left( \frac{\mu_q}{T_c} + \frac{\mu_{crit}}{T_c} \right) \left( \frac{\mu_q}{T_c} - \frac{\mu_{crit}}{T_c} \right) \]

\[ \frac{\partial^2 \ln Z}{\partial \mu^2_q} \sim t^{-\alpha}, \quad \frac{\partial^4 \ln Z}{\partial \mu^4_q} \sim t^{-(2+\alpha)} \quad (\mu > 0) \]

- Already second derivative w.r.t. $\mu_q$ "looks like a specific heat"

\[ \langle (\delta N_q)^2 \rangle \text{ develops a cusp} \]
\[ \langle (\delta N_q)^4 \rangle \text{ diverges on the O(4) critical line;} \]

strength $\sim \left( \frac{\mu_q}{T_c} \right)^4 \quad (\sim 10^{-4} \text{ at RHIC})$
Fluctuations of baryon number and charge densities \((\mu \geq 0)\)

baryon number density fluctuations:

\[
\frac{\chi_B}{T^2} = \left( \frac{d^2}{d(\mu_B/T)^2} \frac{p}{T^4} \right)_{T \text{ fixed}}
\]

\[
= \frac{T}{V} (\langle N_B^2 \rangle - \langle N_B \rangle^2)
\]

susceptibilities

- to be studied in event-by-event fluctuations
- to be compared to hadron resonance gas phenomenology

seeing "true" singular behavior as signal for a critical point requires large volumes and high order Taylor expansions

\[m_\pi = 220 \text{ MeV}, \text{(2+1)-flavor QCD}\]

evidence for a critical point??

\[
\mu_B/T = \{0.0, 1.5, 3.0\}
\]

\[
T \text{ [MeV]}
\]

\[
\chi_B/T^2
\]

\[
\chi_Q/T^2
\]
Isothermal compressibility of the quark gluon plasma

RBC-Bielefeld, preliminary:

\[ N_\tau = 4, 6, m_\pi = 220 \text{ MeV} \]

\[ \mathcal{O}(a^2) \text{ improved} \]

\[ N_\tau = 4, \mathcal{O}(\mu^4) \]

ideal \( q \bar{q} \) gas

inverse compressibility:

\[ \kappa_T^{-1} = \frac{n_B}{\chi_B} = \left( \frac{\partial p}{\partial n_B} \right)_{T \text{ fixed}} \]

high-T, massless limit: polynomial in \( (\mu_B/T)^3 \)

\[ \frac{n_B}{\chi_B} = \mu_B + \mathcal{O} \left( \left( \frac{\mu_B}{T} \right)^3 \right) \]

large density fluctuations for \( \mu_B > 0, T < T_c \)

"saturated" by fluctuations in a hadron resonance gas

resonance gas

expect:

\[ \left( \frac{\partial p}{\partial n_B} \right)_{T} = \frac{n_B}{\chi_B} = 0 \]

at chiral critical point
Conclusions

- Calculations for non-vanishing chemical potential ($\mu_q > 0$) show a rapid transition from a HRG to a QGP; signaled by sudden changes in EoS and susceptibilities.

- Fluctuations on the crossover line increase with increasing baryon chemical potential in accordance with expectations based on a hadron resonance gas model.

- Evidence for additional singular contributions on top of this, which would give direct evidence for the existence of a critical point, is still weak ($O(\mu^6)$ contribution?)

- Need to reduce statistical uncertainty in higher order expansion coefficients to substantiate evidence for the existence of a critical point - within the Taylor expansion approach this is just a question of computing power.

- A large critical value, $\mu_B^{\text{crit}} / T > 2$, is favored.
Pressure, Energy and Entropy

$p/T^4$ from integration over $(\epsilon - 3p)/T^5$;
using piecewise quadratic fit with $T_0 = 100$ MeV with $p(T_0) = 0$;

systematic error on $3p/T^4 \simeq 0.33$
$T = 0$ scale setting using the heavy quark potential

use $r_0$ or string tension to set the scale for $T = 1/N_\tau a(\beta)$

$$V(r) = -\frac{\alpha}{r} + \sigma r$$

$$r^2 \frac{dV(r)}{dr} \bigg|_{r=r_0} = 1.65$$

no significant cut-off dependence when cut-off varies by a factor 5

i.e. from the transition region on $N_\tau = 4$ lattices ($a \simeq 0.25$ fm) to that on $N_\tau = 20$ lattices ($a \simeq 0.05$ fm) !!
scales extracted from 
’gold plated observables’

- high precision studies of several experimentally well known observables in lattice calculations with staggered (asqtad) fermions led to convincing agreement ⇒ gold plated observables
- simultaneous determination of $r_0/a$ in these calculations determines the scale $r_0$ in MeV

knowing any of these experimentally accessible quantities accurately from a lattice calculation is equivalent to knowing $r_0$, which is a fundamental parameter of QCD

A. Gray et al., PRD72 (2005) 094507
high precision studies of several experimentally well known observables in lattice calculations with staggered (asqtad) fermions led to convincing agreement ⇒ gold plated observables

simultaneous determination of $r_0/a$ in these calculations determines the scale $r_0$ in MeV

we use $r_0 = 0.469(7)$ fm determined from quarkonium spectroscopy

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