Perspectives on Pairing in Nuclei

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Fermion Pairing

- A salient property of multi-fermion systems.
- Already in 1950 Meyer suggested that short-range attractive N-N interaction yields J=0 nuclear ground states.
- First microscopic theory of pairing: Bardeen, Cooper, Schriffer (BCS) theory, 1957
- Applications of BCS theory to nuclear structure: Bohr, Belyaev, Migdal 1958-1959 (Application of the BCS theory to nuclear structure has main drawback: BCS wave function is NOT an eigenstate of the number operator).
- Theory of pairing in nuclear physics has many parallels with the theory of ultrasmall metallic grains in condensed matter physics.
Quasi-Spin Algebra

\[ \hat{S}_j^+ = \sum_{m>0} (-1)^{(j-m)} a_{jm}^+ a_{j-m}^+ , \]

\[ \hat{S}_j^- = \sum_{m>0} (-1)^{(j-m)} a_{j-m} a_{jm} , \]

\[ \hat{S}_j^0 = \frac{1}{2} \sum_{m>0} \left( a_{jm}^+ a_{j-m} + a_{j-m}^+ a_{j-m} - 1, \right) \]

Mutually commuting SU(2) algebras:

\[ [\hat{S}_i^+, \hat{S}_j^-] = 2\delta_{ij} \hat{S}_j^0 , \quad [\hat{S}_i^0, \hat{S}_j^\pm] = \pm \delta_{ij} \hat{S}_j^\pm \]
\[ \hat{S}_j^0 = \hat{N}_j - \frac{1}{2} \Omega_j. \]

\( \Omega_j = j + \frac{1}{2} \) = the maximum number of pairs that can occupy the level \( j \)

\[ \hat{N}_j = \frac{1}{2} \sum_{m>0} \left( a_{j m}^\dagger a_{j m} + a_{j-m}^\dagger a_{j-m} \right). \]

0 < \( \hat{N}_j < \Omega_j \) \( \rightarrow \) \( \frac{1}{2} \Omega_j \) representation
Nucleons interacting with a pairing force:

\[ \hat{H} = \sum_{jm} \epsilon_j a_j^\dagger a_j^m - |G| \sum_{jj'} c_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- . \]

In the rest of this talk I will assume that the pairing strength is separable \((c_{jj'} = c_j^* c_{j'})\):

\[ \hat{H} = \sum_{jm} \epsilon_j a_j^\dagger a_j^m - |G| \sum_{jj'} c_j^* c_{j'} \hat{S}_j^+ \hat{S}_{j'}^- . \]
If all occupation probabilities are the same we have

$$\hat{H} = \sum_{jm} \epsilon_j a_{jm}^\dagger a_{jm} - |G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^-.$$

If we further assume that the energy levels are degenerate the first term is a constant for a given number of pairs. This can be solved by using quasispin algebra since $H \propto S^+ S^-$. (Kerman)
Exactly solvable cases:

- Quasi-spin limit (Kerman)
  \[ \hat{H} = -|G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- . \]

- Richardson’s solution:
  \[ \hat{H} = \sum_{jm} \epsilon_j a_j^{\dagger} a_{jm} - |G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- . \]

- Gaudin’s model - closely related to Richardson’s.
  The limit in which the energy levels are degenerate (the first term is a constant for a given number of pairs):
  \[ \hat{H} = -|G| \sum_{jj'} c_j^{\dagger} c_{j'} \hat{S}_j^+ \hat{S}_{j'}^- . \]
  (Draayer, Pan, Balantekin, Pehlivan, de Jesus)

- Most general separable case with two shells (Balantekin and Pehlivan).
Gaudin Algebra

\[
\left[ J^+(\lambda), J^-(\mu) \right] = 2 \frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},
\]

\[
\left[ J^0(\lambda), J^{\pm}(\mu) \right] = \pm \frac{J^{\pm}(\lambda) - J^{\pm}(\mu)}{\lambda - \mu},
\]

\[
\left[ J^0(\lambda), J^0(\mu) \right] = \left[ J^{\pm}(\lambda), J^{\pm}(\mu) \right] = 0
\]

A possible realization:

\[
J^0(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^0}{\epsilon_i - \lambda} \quad \text{and} \quad J^{\pm}(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^{\pm}}{\epsilon_i - \lambda}.
\]
\[ H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda) \]

Not the Casimir operator of the Gaudin algebra!

\[ [H(\lambda), H(\mu)] = 0 \]

Lowest weight vector

\[ J^-(\lambda)|0\rangle = 0, \quad \text{and} \quad J^0(\lambda)|0\rangle = W(\lambda)|0\rangle \]

\[ H(\lambda)|0\rangle = \left[ W(\lambda)^2 - W'(\lambda) \right]|0\rangle \]
How do we find other eigenstates? Consider the state \( |\xi\rangle \equiv J^+(\xi)|0\rangle \) for an arbitrary complex number \( \xi \). Since

\[
[H(\lambda), J^+(\xi)] = \frac{2}{\lambda - \xi} \left( J^+(\lambda)J^0(\xi) - J^+(\xi)J^0(\lambda) \right).
\]

Hence if \( W(\xi) = 0 \), then \( J^+(\xi)|0\rangle \) is an eigenstate of \( H(\lambda) \) with the eigenvalue

\[
E_1(\lambda) = \left[ W(\lambda)^2 - W'(\lambda) \right] - 2 \frac{W(\lambda)}{\lambda - \xi}.
\]

Gaudin showed that this can be generalized.
A state of the form

$$|\xi_1, \xi_2, \ldots, \xi_n \rangle \equiv J^+(\xi_1)J^+(\xi_2)\ldots J^+(\xi_n)|0\rangle$$

is an eigenvector of $H(\lambda)$ if the numbers $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ satisfy the so-called Bethe Ansatz equations:

$$W(\xi_\alpha) = \sum_{\substack{\beta=1 \atop (\beta \neq \alpha)}}^{n} \frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \ldots, n.$$

Corresponding eigenvalue is

$$E_n(\lambda) = \left[ W(\lambda)^2 - W'(\lambda) \right] - 2 \sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}.$$
Recall the Gaudin Algebra

\[
[J^+(\lambda), J^-(\mu)] = 2 \frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},
\]

\[
[J^0(\lambda), J^{\pm}(\mu)] = \pm \frac{J^{\pm}(\lambda) - J^{\pm}(\mu)}{\lambda - \mu},
\]

\[
[J^0(\lambda), J^0(\mu)] = [J^{\pm}(\lambda), J^{\pm}(\mu)] = 0
\]

Not only the operators \( J(\lambda) \), but also the operators \( J(\lambda) + c \) satisfy this algebra for a constant \( c \). In this case

\[
H(\lambda) = J(\lambda) \cdot J(\lambda) \Rightarrow H(\lambda) + 2c \cdot J(\lambda) + c^2
\]

which has the same eigenstates.
Richardson operators:

\[
\lim_{\lambda \to \epsilon_k} (\lambda - \epsilon_k) (H(\lambda) + 2c \cdot S) = R_k
\]

\[
R_k = -2c \cdot S_k - 2 \sum_{j \neq k} \frac{S_k \cdot S_j}{\epsilon_k - \epsilon_j}
\]

\[
[H(\lambda) + 2c \cdot S, R_k] = 0 \quad [R_j, R_k] = 0
\]
\[ \hat{H} = \sum_{jm} \epsilon_j a^\dagger_{jm} a_{jm} - |G| \sum_{jj'} \hat{S}^+_j \hat{S}^-_{j'} . \]

\[ \Rightarrow H = \sum_j \epsilon_j S^0_j - |G| \left( \left( \sum_i S_i \right) \cdot \left( \sum_i S_i \right) - \left( \sum_i S^0_i \right)^2 + \left( \sum_i S_i \right) \right) \]

+ constant terms

Choose

\[ \mathbf{c} = (0, 0, -1/2|G|) \]

then

\[ \frac{H}{|G|} = \sum_i \epsilon_i R_i + |G|^2 \left( \sum_i R_i \right)^2 - |G| \sum_i R_i + \cdots \]
Degenerate Solution

Define

\[ \hat{S}^+(0) = \sum_j c_j^* \hat{S}^+_j \quad \text{and} \quad \hat{S}^-(0) = \sum_j c_j \hat{S}^+_j, \]

\[ \hat{H} = -|G| \hat{S}^+(0) \hat{S}^-(0). \]

A state of the form

\[ \hat{S}^+(0)|0\rangle = \sum_j c_j^* \hat{S}^+_j|0\rangle, \quad |0\rangle: \text{particle vacuum} \]

is an eigenstate:

\[ \hat{H}\hat{S}^+(0)|0\rangle = \left( -|G| \sum_j \Omega_j |c_j|^2 \right) \hat{S}^+(0)|0\rangle \]

Are there other one-pair states?
| Energy/\(-|G|\) | State |
|-----------------|--------|
| 0               | \(\left(-\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}^+_{j_1} + \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}^+_{j_2}\right)|0\) |
| \(\Omega_{j_1}|c_{j_1}|^2 + \Omega_{j_2}|c_{j_2}|^2\) | \(\left(c^*_{j_1} \hat{S}^+_{j_1} + c^*_{j_2} \hat{S}^+_{j_2}\right)|0\) |

States with \(N=1\) for two shells
Is there a systematic way to derive these states? Yes, as showed by Pan, et al. for particle pair states.

Define

\[ \hat{S}^+(x) = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}^+_j \quad \text{and} \quad \hat{S}^-(x) = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}^-_j. \]

Then eigenstates are of the form

\[ \hat{S}^+(x) \hat{S}^+(y) \cdots \hat{S}^+(z) |0\rangle \]

\[
\hat{S}^+_j = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}^+_j
\]

and
\[
\hat{S}^-_j = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}^-_j
\]

Introduce the operator
\[
\hat{K}^0(x) = \sum_j \frac{1}{1/|c_j|^2 - x} \hat{S}^0_j
\]

\[
[\hat{S}^+(x), \hat{S}^-(0)] = [\hat{S}^+(0), \hat{S}^-(x)] = 2\hat{K}^0(x)
\]

\[
[\hat{K}^0(x), \hat{S}^\pm(y)] = \pm \frac{\hat{S}^\pm(x) - \hat{S}^\pm(y)}{x - y}
\]

This is very similar to Gaudin algebra!
\( \hat{S}^+(0) \hat{S}^+(z_1^{(N)}) \ldots \hat{S}^+(z_{N-1}^{(N)}) |0\rangle \)

is an eigenstate if the following Bethe ansatz equations are satisfied:

\[
\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - Z_m^{(N)}} = \frac{1}{Z_m^{(N)}} + \sum_{k=1, k\neq m}^{N-1} \frac{1}{Z_m^{(N)} - Z_k^{(N)}} \quad m = 1, 2, \ldots N-1.
\]

\[
E_N = -|G| \left( \sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{Z_k^{(N)}} \right)
\]

Pan et al did not note but this is an eigenstate if the shell is at most half full.
Similarly
\[ \hat{\mathcal{S}}^+ (x_1^{(N)}) \hat{\mathcal{S}}^+ (x_2^{(N)}) \ldots \hat{\mathcal{S}}^+ (x_N^{(N)}) |0\rangle \]
is an eigenstate with zero energy if the following Bethe ansatz equations are satisfied:

\[ \sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - x_m^{(N)}} = \sum_{k=1(k\neq m)}^N \frac{1}{x_m^{(N)} - x_k^{(N)}} \quad \text{for every } m = 1, 2, \ldots, N \]

Again this is a state if the shell is at most half full.
What if the available states are more than half full? There are degeneracies:

| No. of Pairs | Energy/$(-|G|)$ | State       |
|--------------|------------------|-------------|
| 1            | $\sum_j \Omega_j |c_j|^2$       | $\hat{S}^+(0)|0\rangle$ |
| $N_{max}$    | $\sum_j \Omega_j |c_j|^2$       | $|\bar{0}\rangle$       |

$|0\rangle$: particle vacuum

$|\bar{0}\rangle$: state where all levels are completely filled
If the shells are more than half full then the state

$$\hat{S}^{-}(z_1^{(N)}) \hat{S}^{-}(z_2^{(N)}) \cdots \hat{S}^{-}(z_{N-1}^{(N)}) |\bar{0}\rangle$$

is an eigenstate with energy

$$E = -G \left( \sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}} \right)$$

if the following Bethe ansatz equations are satisfied

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1}^{N-1} \frac{1}{z_m^{(N)} - z_k^{(N)}}$$

Here $N_{max} + 1 - N = \text{number of particle pairs}$

Particle-hole degeneracy:

<table>
<thead>
<tr>
<th>No. of Pairs</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$\hat{S}^+(0)\hat{S}^+(z_1^{(N)}) \ldots \hat{S}^+(z_{N-1}^{(N)})</td>
</tr>
<tr>
<td>$N_{\text{max}} + 1 - N$</td>
<td>$\hat{S}^-(z_1^{(N)})\hat{S}^-(z_2^{(N)}) \ldots \hat{S}^-(z_{N-1}^{(N)})</td>
</tr>
</tbody>
</table>

$$E = -G \left( \sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{Z_k^{(N)}} \right)$$

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - Z_m^{(N)}} = \frac{1}{Z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{Z_m^{(N)} - Z_k^{(N)}}$$
Results for the sd shell with $0d_{5/2}$, $0d_{3/2}$, and $1s_{1/2}$

<table>
<thead>
<tr>
<th></th>
<th>$^{58}$Ni</th>
<th>$^{60}$Ni</th>
<th>$^{62}$Ni</th>
<th>$^{64}$Ni</th>
<th>$^{66}$Ni</th>
<th>$^{68}$Ni</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-0.33$</td>
<td>$-0.33$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.73</td>
<td>$-1.40$</td>
<td>$-1.40$</td>
<td>1.73</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.94</td>
<td>2.63</td>
<td>$-2.40$</td>
<td>$-2.40$</td>
<td>2.63</td>
<td>2.94</td>
<td></td>
</tr>
<tr>
<td>1 Neutron Pair</td>
<td>$-4.73$</td>
<td>$-3.92$</td>
<td>$-3.92$</td>
<td>$-4.73$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Neutron Pairs</td>
<td>$-5.56$</td>
<td>$-5.56$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Neutron Pairs</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Hole Pairs</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1 Hole Pair</td>
<td></td>
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<tr>
<td>0 Hole Pairs (Full shell)</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Exact solutions with two shells

Consider the most general pairing Hamiltonian with only two shells:

\[
\hat{H}_{\text{pair}} = \frac{\hat{H}}{|G|} = \sum_j 2\epsilon_j \hat{S}_j^0 - \sum_{jj'} c^*_j c_{j'} \hat{S}_j^+ \hat{S}_{j'}^- + \sum_j \epsilon_j \Omega_j,
\]

with \( \epsilon_j = \epsilon_j/|G| \).

States can be written using the step operators

\[
J^+(x) = \sum_j \frac{c^*_j}{2\epsilon_j - |c_j|^2 x} S_j^+
\]

as

\[
J^+(x_1) J^+(x_2) \ldots J^+(x_N) |0\rangle.
\]

\[ J^+(x_1) J^+(x_2) \ldots J^+(x_N) |0\rangle. \]

Defining

\[ \beta = 2 \frac{\varepsilon_{j_1} - \varepsilon_{j_2}}{|c_{j_1}|^2 - |c_{j_2}|^2} \quad \delta = 2 \frac{\varepsilon_{j_2} |c_{j_1}|^2 - \varepsilon_{j_1} |c_{j_2}|^2}{|c_{j_1}|^2 - |c_{j_2}|^2}. \]

we obtain

\[ E_N = - \sum_{n=1}^{N} \frac{\delta x_n}{\beta - x_n}. \]

If the parameters \( x_k \) satisfy the Bethe ansatz equations

\[ \sum_j \frac{\Omega_j |c_j|^2}{2 \varepsilon_j - |c_j|^2 x_k} = \frac{\beta}{\beta - x_k} + \sum_{n=1(\neq k)}^{N} \frac{2}{x_n - x_k}. \]
Exact Energy eigenvalues for $j_1 = 3/2$ and $j_2 = 5/2$. $\cos \vartheta = c_1$ and $\sin \vartheta = c_2$ $\Delta = \epsilon_1 - \epsilon_2$. 
Solutions of Bethe Ansatz equations

\[ x_i^{(N)} = \frac{1}{|c_{j_2}|^2} + \eta_i^{(N)} \left( \frac{1}{|c_{j_1}|^2} - \frac{1}{|c_{j_2}|^2} \right) \]

\[ \sum_{k=1, k \neq i}^{N} \frac{1}{\eta_i^{(N)} - \eta_k^{(N)}} - \frac{\Omega_{j_2}/2}{\eta_i^{(N)}} + \frac{\Omega_{j_1}/2}{1 - \eta_i^{(N)}} = 0 \]

In 1914 Stieltjes showed that the polynomial

\[ p_N(z) = \prod_{i=1}^{N} (z - \eta_i^{(N)}) \]

satisfies the hypergeometric equation

\[ z(1-z)p''_N + \left[ -\Omega_{j_2} + (\Omega_{j_1} \Omega_{j_2}) z \right] p'_N + N \left( N - \Omega_{j_1} - \Omega_{j_2} - 1 \right) p_N = 0 \]
Supersymmetric Quantum Mechanics

Consider two Hamiltonians

\[ H_1 = G^\dagger G, \quad H_2 = GG^\dagger, \]

where \( G \) is an arbitrary operator. The eigenvalues of these two Hamiltonians

\[ G^\dagger G|1, n\rangle = E_n^{(1)}|1, n\rangle \]
\[ GG^\dagger|2, n\rangle = E_n^{(2)}|2, n\rangle \]

are the same:

\[ E_n^{(1)} = E_n^{(2)} = E_n \]

and that the eigenvectors are related:

\[ |2, n\rangle = G \left[ G^\dagger G \right]^{-1/2} |1, n\rangle. \]

This works for all cases except when \( G|1, n\rangle = 0 \), which should be the ground state energy of the positive-definite Hamiltonian \( H_1 \).
Why is this called supersymmetry? Define

\[ Q^\dagger = \begin{pmatrix} 0 & 0 \\ G^\dagger & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}, \]

Then

\[ H = \{ Q, Q^\dagger \} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}. \]

with

\[ [H, Q] = 0 = [H, Q^\dagger]. \]
SUSY QM in Nuclear Pairing

Separable pairing with degenerate single-particle spectra:

\[ \hat{H}_{SC} \sim -|G| \hat{S}^+(0) \hat{S}^-(0), \]

\[ \hat{S}^+(0) = \sum_j c_j^* \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(0) = \sum_j c_j \hat{S}_j^- . \]

Introduce the operator

\[ \hat{T} = \exp \left( -i \frac{\pi}{2} \sum_i (\hat{S}^+_i + \hat{S}^-_i) \right) \]

This operator transforms the empty shell, \( |0\rangle \), to the fully occupied shell, \( |\bar{0}\rangle \):

\[ \hat{T}|0\rangle = |\bar{0}\rangle \]

Next define

\[ \hat{B}^- = \hat{T}^\dagger \hat{S}^-(0), \quad \hat{B}^+ = \hat{S}^+(0) \hat{T} . \]
Supersymmetric quantum mechanics tells us that the partner Hamiltonians $\hat{H}_1 = \hat{B}^+ \hat{B}^-$ and $\hat{H}_2 = \hat{B}^- \hat{B}^+$ have identical spectra except for the ground state of $\hat{H}_1$.

Here two Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ are actually identical and equal to the pairing Hamiltonian. Hence the role of the supersymmetry is to connect the states $|\psi_2\rangle$ and $|\psi_1\rangle$.

This supersymmetry connects particle and hole states.

Spectra of Nuclear pairing exhibiting supersymmetry