Quantum Strings in $AdS_5 \times S^5$

Arkady Tseytlin

- Recent progress in perturbative GS superstring:
  first 2-loop computation and nontrivial check of BES ansatz

- Towards solution of $AdS_5 \times S^5$ superstring:
  reformulation in terms of physical d.o.f.
  \textbf{M. Grigoriev and A.T., arXiv:0711.0155}
  \textbf{R. Roiban and A.T., to appear}
$\mathcal{N} = 4$ SYM at $N = \infty$
dual to type IIB superstrings in $AdS_5 \times S^5$

$\lambda = g_{YM}^2 N$ \quad \text{related to string tension}

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \rightarrow 0$

need to go beyond BPS states and
“supergravity + classical probes” approximation
Problems:

- spectrum of states (exact energies in $\lambda$)
- construction of vertex operators (closed and open string ones)
- computation of their correlation functions (graviton scattering, application to DIS in QCD ?)
- expectation values of various Wilson loops
- gluon scattering amplitudes
- generalizations to simplest less supersymmetric cases
  - orbifolds, exactly marginal deformations, type 0 string in $AdS_5 \times S^5$
- strings at finite temperature in $AdS_5 \times S^5$
  (without black hole and with it ...)
- non-critical superstrings: $AdS_5 \times S^1$, ...
Type IIB strings in $AdS_5 \times S^5$

Recent remarkable progress in quantitative understanding interpolation from weak to strong ‘t Hooft coupling based on using perturbative gauge theory (4-loop in $\lambda$) and perturbative string theory (2-loop in $\frac{1}{\sqrt{\lambda}}$) “data” and assumption of exact integrability

string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, \ldots) = \Delta(\lambda, J, m, \ldots)$$

$J$ - charges of $SO(2, 4) \times SO(6)$:

spins $S_1, S_2$; $J_1, J_2, J_3$

$m$ - windings, folds, cusps, oscillation numbers, ...

Operators: $\text{Tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_-^{S_2} \ldots F_{mn} \ldots \Psi \ldots)$

Solve susy 4-d CFT = string in R-R background:
compute $E = \Delta$ for any $\lambda$ (and $J, m$)
Perturbative expansions are opposite:
\( \lambda \gg 1 \) in perturbative string theory
\( \lambda \ll 1 \) in perturbative planar gauge theory
use perturbative results on both sides
and other properties (integrability, susy,...)
to come up with an exact answer – Bethe ansatz
Last 5 years: remarkable progress:
“semiclassical” string states with large quantum numbers
dual to “long” gauge operators (BMN, GKP, ...)
\( E = \Delta \) – same dependence on \( J, m, ... \)
coefficients = interpolating functions of \( \lambda \)

SYM dilatation operator that determines \( \Delta \)
is same as an integrable spin chain Hamiltonian;
integrability at both perturbative gauge (\( \lambda \ll 1 \))
and string (\( \lambda \gg 1 \)) sides;
\( \rightarrow \) suggests Bethe ansatz for the spectrum at any \( \lambda \)
Heisenberg-model type BA
(Beisert, Dippel, Staudacher 04; Staudacher 05)

\[ e^{\imath p_k J} = \prod_{j \neq k} S(p_k, p_j; \lambda), \quad S = S_1 e^{\imath \theta} \]

\[ S_1 = \frac{u_k - u_j + i}{u_k - u_j - i}, \quad \theta = \theta(p_k, p_j; \lambda) \]

scattering of elementary excitations (magnons)
with 1-d momenta \( p_j \) and rapidities \( u_j \)

\[ u_j(p_j, \lambda) = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} \]

\[ E = J + \sum_{j=1}^{M} \left( \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} - 1 \right) \]

What about phase \( \theta \)?
structure fixed by symmetries (Beisert 05)

\[ \theta(p, p'; \lambda) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{rs}(\lambda) [q_s(p') q_r(p) - q_s(p) q_r(p')] \]
\[
q_{r+1}(p) = \frac{2}{r} \sin \frac{rp}{2} \left( \frac{\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} - 1}{\frac{\lambda}{\pi^2} \sin \frac{p}{2}} \right)^r,
\]

\[c_{rs}(\lambda) =?\]
crucial input from string theory:
\[
c_{rs}(\lambda \gg 1) = \lambda^{\frac{r+s-1}{2}} \left[ \delta_{r,s-1} + \frac{1}{\sqrt{\lambda}} a_{rs} + \frac{1}{(\sqrt{\lambda})^2} b_{rs} + \ldots \right]
\]
String 1-loop corrections to string energies
(Frolov, AT 03; Park, Tirziu, AT 05) → \( a_{rs} \neq 0 \) (Beisert, AT 05)
1-loop string results translate into (Hernandez, Lopez 06)
\[
a_{rs} = \frac{2}{\pi} \left[ 1 - (-1)^{r+s} \right] \frac{(r-1)(s-1)}{(r-1)^2 - (s-1)^2}
\]
Consistent (Arutyunov, Frolov 06; Beisert 06)
with “crossing” (Janik 06)
All-order guess for strong coupling expansion
(Beisert, Hernandez, Lopez 06)

Finally fixed completely (Beisert, Eden, Staudacher 06)
as weak-coupling expansion matching 4-loop result of Bern et al
But first-principles derivation remains to be given

Problem:
solve string theory in $AdS_5 \times S^5$
in particular, on an infinite line →
determine the magnon (BMN excitation) scattering S-matrix →
derive BA with the right BHL/BES phase
String Theory in $AdS_5 \times S^5$

bosonic coset $AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$ (Metsaev, AT 98)

$$S = T \int d^2 \sigma \left[ G_{mn}(x) \partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta \partial x \\
+ \bar{\theta} \theta \bar{\theta} \theta \partial x \partial x + \ldots \right]$$

tension $T = \frac{R^2}{2\pi \alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset $\sigma$-model (Luscher-Pohlmeyer 76)

Applies to $AdS_5 \times S^5$ superstring (Bena, Polchinski, Roiban 02)

Progress in understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04; Beisert et al 05; Dorey, Vicedo 06,...)
Explicit computation of 1-loop quantum superstring corrections to classical string energies (Frolov, AT 02-4, ...) results were used as input for 1-loop term in strong-coupling expansion of the phase $\theta$ in BA

Tree-level S-matrix of BMN states from $AdS_5 \times S^5$ GS string agrees with limit of elementary magnon S-matrix (Klose, McLoughlin, Roiban, Zarembo 06)

Semiclassical S-matrix in different limits: string solitons on an infinite line – Giant magnons (Hofman, Maldacena 06; Dorey 06, ...) “Near-flat” limit (Hofman, Maldacena 07) studied at 1-loop level with consistent results...
Last year:

2-loop string corrections (Roiban, Tirziu, AT; Roiban, AT 07)
2-loop check of finiteness of the GS superstring;
agreement with BA
– implicit check of integrability of quantum string theory
– non-trivial confirmation of BES exact phase in BA
– comparison to strong-coupling expansion
of BES equation (Basso, Korchemsky, Kotansky 07)
should extend to higher loop level
Universal scaling function =
Cusp anomalous dimension

gauge theory:  \( \text{Tr}(\Phi D^S_{\pm} \Phi) \)

\[ \Delta = S + 2 + f(\lambda) \ln S + ... , \quad S \gg 1 \]

\[ f(\lambda \ll 1) = c_1 \lambda + c_2 \lambda^2 + c_3 \lambda^3 + c_4 \lambda^4 + ... \]

c\(_n\) are given by Feynmann graphs of 4d CFT – N=4 SYM

string theory:  rotating folded string with spin \( S \) in \( AdS_5 \)

\[ f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi} [a_0 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + ... ] \]

a\(_n\) are given by Feynmann graphs of 2d CFT – \( AdS_5 \times S^5 \) string
Explicitly:

\[
f_{\lambda \ll 1} = \frac{1}{2\pi^2} \left[ \lambda - \frac{\lambda^2}{48} + \frac{11\lambda^3}{2^8 \times 45} - \left( \frac{73}{630} + \frac{4(\zeta(3))^2}{\pi^6} \right) \frac{\lambda^4}{2^7} + \ldots \right]
\]

\[
c_3: \text{Kotikov, Lipatov, et al 03; } c_4: \text{Bern, Dixon, et al 06}
\]

\[
f_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi} \left[ 1 - \frac{3 \log 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} + \ldots \right]
\]

\[
a_0: \text{Gubser, Klebanov, Polyakov 02; } a_1: \text{Frolov, AT 02}
\]

\[
a_2: \text{Roiban, AT 07}
\]

\[
K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915\ldots - \text{Catalan’s constant}
\]

appears from 2-loop sigma model integrals

Smooth interpolation from weak to strong coupling
Remarkably, both expansions are reproduced from single Beisert-Eden-Staudacher integral equation for $f(\lambda)$ obtained using the exact BES phase in the BA strong-coupling expansion of BES eq.: numerical results: Benna, Benvenutti, Klebanov, Scardicchio 07 analytic expansion: Basso, Korchemsky, Kotansky 07 in full agreement with string theory $\rightarrow$ 2-loop string check of the BES phase

BES equation: $f(\lambda)$ known at least in principle to any order in small $\lambda$ and large $\lambda$ expansion

**Exact solution** hopefully will be found ...
weak coupling (BES): \( f(\lambda) = \sum_{n=1}^{\infty} c_n (\frac{\lambda}{4\pi^2})^n \)

\[ c_1 = 2, \quad c_2 = -\zeta_2, \quad c_3 = 88\zeta_4, \quad c_4 = -16(73\zeta_6 + 4\zeta_3^2) \]
\[ c_5 = 32(887\zeta_8 + 8\zeta_2\zeta_3^2 + 40\zeta_3\zeta_5) \]
\[ c_6 = -64(136883\zeta_{10} + 8\zeta_2\zeta_3^2 + 80\zeta_2\zeta_3\zeta_5 + 210\zeta_3\zeta_7 + 102\zeta_5^2), \quad \ldots \]

strong coupling (BKK): \( f(\lambda) = \frac{\sqrt{\lambda}}{\pi} \sum_{n=0}^{\infty} \frac{a_n}{(\sqrt{\lambda})^n} \)

\[ a_0 = 1, \quad a_1 = -3\zeta_1, \quad a_2 = -\beta_2, \quad a_3 = -\frac{1}{32} \left[ 27\zeta_3 + 96\beta_2\zeta_1 \right] \]
\[ a_4 = -\frac{1}{16} \left[ 84\beta_4 + 81\zeta_3\zeta_1 + 32\beta_2^2 + 144\beta_2\zeta_1^2 \right], \]
\[ a_5 = -\frac{9}{2048} \left[ 4785\zeta_5 + 10572\beta_4\zeta_1 + 4416\zeta_3\beta_2 + 5184\zeta_3\zeta_1^2 + 4096\beta_2^2\zeta_1 \right], \quad \ldots \]

\[ \zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}, \quad \zeta_{2n} \sim \pi^{2n}, \quad \zeta_1 \equiv -\sum \frac{(-1)^n}{n} = \ln 2, \]
\[ \beta_k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k}, \quad \beta_2 = K, \quad \beta_{2n+1} \sim \pi^{2n+1} \]

non-trivial function in 4d QFT
Beyond 2-loop order in string theory?
Deeper understanding of quantum string theory from integrability point of view?
Exact string S-matrix?
Proof of the BES Bethe ansatz?
Green-Schwarz superstring in $AdS_5 \times S^5$

Superstring in curved type II supergravity background

$$\int d^2 \sigma \ G_{MN}(Z) \partial Z^M \partial Z^N + \ldots$$

$$Z^M = (x^m, \theta^I_\alpha)$$

$$m = 0, 1, \ldots, 9, \quad \alpha = 1, 2, \ldots, 16, \quad I = 1, 2$$

Explicit form of action is generally hard to find

$AdS_5 \times S^5$: coset space symmetry facilitates explicit construction

Algebraic construction of unique $\kappa$-invariant action as in flat space

GS superstring in flat space:

$$R^{1,9} = \frac{G}{H} = \frac{\text{Poincare}}{\text{Lorentz}}$$

Flat superspace = $\frac{\hat{G}}{\hat{H}} = \frac{\text{SuperPoincare}}{\text{Lorentz}}$

structure of action is fixed by superPoincare algebra $(P, M, Q)$

$$[P, M] \sim P, \quad [M, M] \sim M, \quad [M, Q] \sim Q, \quad \{Q, Q\} \sim P$$

$$g^{-1} dg = J^m P_m + J^I_\alpha Q^\alpha_I + J^{mn} M_{mn}$$

Supercoset action = $\int \text{Tr}(g^{-1} dg)^2_{G/H}$ + fermionic WZ-term

$$I = \int d^2 \sigma (J^m J^m + a J^I J^I) + b \int J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}$$

$s_{IJ} = (1, -1)$
Manifest superPoincare symmetry, but unitarity and right fermionic spectrum iff $a = 0, \ b = \pm 1$:
\[\kappa\text{-invariance} \rightarrow \text{Green-Schwarz action:}\]

\[
L = -\frac{1}{2}(\partial_a x^m - i\bar{\theta}^I \Gamma^m \partial_a \theta^I)^2 \\
+ i\epsilon^{ab} s_{IJ} \bar{\theta}^I \Gamma_m \partial_a \theta^J (\partial_b x^m - i\bar{\theta}^K \Gamma^m \partial_b \theta^K)
\]

peculiar "degenerate" Lagrangian: no $\partial \bar{\theta} \partial \theta$ term

$L \sim \partial x \partial x + \partial x \bar{\theta} \partial \theta + (\bar{\theta} \partial \theta)^2$

perturbative expansion is well-defined near $\bar{x}$ background, e.g., $x^m = N_a^m \sigma^a$

$x = \bar{x} + \xi, \ \theta' = \sqrt{\partial \bar{x}} \ \theta$

$L \sim \partial \xi \partial \xi + \bar{\theta}' \partial \theta' + \frac{1}{\sqrt{\partial \bar{x}}} \partial \xi \bar{\theta}' \partial \theta' + ...$

non-renormalizable by power counting

but $\kappa$-symmetry (uniqueness of action) implies finiteness
direct check of cancellation of 2-loop logarithmic UV divergences and trivial partition function (Roiban, Tirziu, AT 07)
preservation of $\kappa$-symmetry implies that semiclassical loop ($\alpha'$) expansion must be finite also in curved space
but regularization issues are non-trivial starting with 2 loops
\[ \text{AdS}_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)} \]

Killing vectors and Killing spinors of \( \text{AdS}_5 \times S^5 \):

\( PSU(2,2|4) \) symmetry

replace \( \frac{G}{H} \) =SuperPoincare/Lorentz in flat GS case by

\[ \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)} \]

generators: \( (P_q, M_{pq}); (P'_r, M'_{rs}); Q^I_\alpha, \ m = (q,r) \)

\[
\begin{align*}
[P, P] & \sim M, \quad [P, M] \sim P, \quad [M, M] \sim M, \\
[Q, P_q] & \sim \gamma_q Q, \quad [Q, M_{pq}] \sim \gamma_{pq} Q \\
\{Q^I, Q^J\} & \sim \delta^{IJ} (\gamma \cdot P + \gamma' \cdot P') + \epsilon^{IJ} (\gamma \cdot M + \gamma' \cdot M')
\end{align*}
\]
**PSU(2, 2|4) invariant action:**

\[
\sqrt{\frac{\lambda}{2\pi}} \left[ \int d^2\sigma (J^m J^m + a \bar{J}^I J^I) + b \int J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ} \right]
\]

as in flat space \( a = 0, \ b = \pm 1 \) required by \( \kappa \)-symmetry

unique action with right symmetry and right flat-space limit

Formal argument for UV finiteness (2d conformal invariance):

1. global symmatry – only overall coefficient
   of \( J^2 \) term (radius) can run

2. non-renormalization of WZ term (homogeneous 3-form)

3. preservation of \( \kappa \)-symmetry at the quantum level
   – relating coefficients of \( J^2 \) and WZ terms
Component form:

coset representative $g(x, \theta) = f(x)e^{\theta Q}$

$J^m = e^m(x) - i\bar{\theta}^I \Gamma^m D\theta^I + O(\theta^4)$, $J^I = D\theta^I + O(\theta^3)$

solving Maurer-Cartan eqs:

$J^A_a = \partial_a x^m e^A_m - 4i\bar{\theta}^I \Gamma^A \frac{\sinh^2(\frac{s}{2}\mathcal{M})}{\mathcal{M}^2} I_J D_a \theta^J$, $J^I_a = \left[ \frac{\sinh(s\mathcal{M})}{\mathcal{M}} D_a \theta \right]^I$, 

$D\theta^I = \mathcal{D}\theta^I - \frac{i}{2} \epsilon^{IJ} e^A(x) \Gamma_\ast \Gamma^A \theta^J$, $\mathcal{D}\theta^I = d\theta^I + \frac{1}{4} \omega^{AB}(x) \Gamma_{AB} \theta^I$,

$(\mathcal{M}^2)^IL = -\epsilon^{IJ} \Gamma_\ast \Gamma^A \theta^J \bar{\theta}^L \Gamma_A + \frac{1}{2} \epsilon^{LK} (\Gamma^{pq} \theta^I \bar{\theta}^K \Gamma_{pq} \Gamma_\ast - \Gamma^{rs} \theta^I \bar{\theta}^K \Gamma_{rs} \Gamma_\ast)$

$e^A(x) = dx^m e^A_m(x)$, $A = (p, r)$

$\Gamma_\ast = i\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$, $\Gamma'_\ast = i\Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9$

RR coupling: “mass term” in $D$

$D$ in IIB Killing spinor eq. $D^{IJ} \epsilon^J = 0$, $[D_M, D_N] = 0$
Expansion near string soliton solution \( x = \bar{x} \):

conformal gauge and \( \kappa \)-symmetry gauge \( \theta^1 = \theta^2 \)

\[
I = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \left( L_{\text{kin}} + L_{\text{WZ}} \right)
\]

\[
L_{\text{kin}} = -\frac{1}{2} \partial_a x^\mu \partial^a x^\nu G_{\mu\nu}(x) + 2ie_a^A \bar{\theta} \Gamma_A \mathcal{D}^a \theta + 2\bar{\theta} \Gamma^A \mathcal{D}_a \theta \bar{\theta} \Gamma_A \mathcal{D}^a \theta \\
+ \frac{1}{12} e_a^A e_{aB} \bar{\theta} \Gamma_A (\Gamma^{pq} \theta \bar{\theta} \Gamma_{pq} - \Gamma^{rs} \theta \bar{\theta} \Gamma_{rs}) \Gamma_B \theta + O(\theta^6)
\]

\[
L_{\text{WZ}} = \epsilon^{ab} \left[ -e_a^A e_b^B \bar{\theta} \Gamma_A \Gamma^* \Gamma_B \theta + \frac{4i}{3} e_a^A \bar{\theta} \Gamma_A \Gamma^* \Gamma_B \theta \bar{\theta} \Gamma^B \mathcal{D}_b \theta \right] + O(\theta^6)
\]

Expansion: \( x \rightarrow x + \xi \), \( L = \xi D^2 \xi + \bar{\theta} D \theta + \xi^3 + \xi^4 + \xi \theta^2 + \theta^4 + \ldots \)

1-loop results:

- check of finiteness of GS action for generic \( \bar{x} \) solution
- computation of 1-loop quantum string corrections to energies of rigid rotating string solutions (Frolov, AT 02,03; Park, AT 05)
  – data for reconstructing 1-loop term in strong-coupling expansion of phase in BA (Beisert, AT 05; Hernandez, Lopez 06)
Simple form of the $AdS_5 \times S^5$ action

special choice of coordinates (Poincare) and special $\kappa$-symmetry gauge: $\theta^1 = \Gamma_{0123}\theta^2$

plus “Killing spinor” redefn of fermions (Kallosh, Rajaraman 98)

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[ z^2 (\partial_a x^m - i \theta^m \partial_a \theta)^2 + \frac{1}{z^2} \partial^a z^s \partial_a z^s + 4 \epsilon^{ab} \bar{\theta} \partial_a z^s \Gamma_s \partial_b \theta \right]$$

$m = 0, 1, 2, 3; \ s = 4, \ldots, 9, \ z^2 = z^s z^s, \ a, b = 0, 1$

after formal T-duality (on $R \times R$): $x^m \rightarrow \tilde{x}^m$

action becomes exactly quadratic in $\theta$ (Kallosh, AT 98)

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[ \frac{1}{z^2} (\partial^a x^m \partial_a x_m + \partial^a z^s \partial_a z^s) + 4 \epsilon^{ab} \bar{\theta} (\partial_a x^m \Gamma_m + \partial_a z^s \Gamma_s) \partial_b \theta \right]$$

starting point of computation of 2-loop string correction to cusp anomalous dimension (Roiban, AT 07)

check of 2-loop finiteness of $AdS_5 \times S^5$ GS string

check of BES phase proposal against 2-loop string theory
How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

not of known solvable type (cf. free oscillators; WZW)

analogy with exact solution of $O(n)$ model (Zamolodchikovs) or
principal chiral model (Polyakov-Wiegmann; KWZ; ...)?

– 2d CFT – no mass generation

Try as in flat space –
light-cone gauge: analog of $x^+ = p^+ \tau$, $p^+ = $ const, $\Gamma^+ \theta = 0$

Two natural options:
(i) null geodesic parallel to the boundary in Poincare patch –
action/Hamiltonian quartic in fermions (Metsaev, Thorn, AT, 01)
(ii) null geodesic wrapping $S^5$:
hidden $su(2|2) \times su(2|2)$ symmetry
but complicated action (Callan et al, 03; Arutyunov, Frolov, Plefka, Zamaklar, 05-06)
Common problem:

**lack of manifest 2d Lorentz symmetry**

hard to apply methods of 2d integrable field theory – 
S-matrix depends on both rapidities, not on their difference; 
constraints on it are unclear, etc.

An alternative approach – **“Pohlmeyer reduction”**
use conf. gauge, solve Virasoro conditions in terms of currents, 
find “reduced” action for physical d.o.f., 
use it as a starting point for quantization 
(Grigoriev, AT 07; Mikhailov, Schafer-Nameki 07)
PR formulation for $AdS_5 \times S^5$ superstring:
(i) introduce new fields locally related to supercoset currents
(ii) solve conformal gauge (Virasoro) condition explicitly
(iii) find local 2d Lorentz-invariant
action for independent (8B+8F) d.o.f. →
fermionic generalization of non-abelian Toda theory

PR: a nonlocal map that preserves integrable structure
1. gauge-equivalent Lax pairs; map between soliton solutions
gives integrable massive local field theory
2. quantum equivalence to original GS model ?
may expect for full $AdS_5 \times S^5$ string model = CFT
3. integrable theory: semiclassical solitonic spectrum
may essentially determine quantum spectrum
the two solitonic S-matrices should be closely related:
Lorentz-invariant S-matrix of PR-model should effectively
give the complicated magnon S-matrix
Pohlmeyer reduction: bosonic coset models

Prototypical example: $S^2$-sigma model $\rightarrow$ Sine-Gordon theory

\[ L = \partial_+ X^m \partial_- X^m - \Lambda (X^m X^m - 1), \quad m = 1, 2, 3 \]

Equations of motion:

\[ \partial_+ \partial_- X^m + \Lambda X^m = 0, \quad \Lambda = \partial_+ X^m \partial_- X^m, \quad X^m X^m = 1 \]

Stress tensor:

\[ T_{\pm\pm} = \partial_{\pm} X^m \partial_{\pm} X^m \]

\[ T_{+-} = 0, \quad \partial_+ T_{-} = 0, \quad \partial_- T_{++} = 0 \]

implies $T_{++} = f(\sigma_+), \quad T_{--} = h(\sigma_-)$

using the conformal transformations $\sigma_{\pm} \rightarrow F_{\pm}(\sigma_{\pm})$ can set

\[ \partial_+ X^m \partial_+ X^m = \mu^2, \quad \partial_- X^m \partial_- X^m = \mu^2, \quad \mu = \text{const} \]

3 unit vectors in 3-dimensional Euclidean space:

\[ X^m, \quad X^m_+ = \mu^{-1} \partial_+ X^m, \quad X^m_- = \mu^{-1} \partial_- X^m, \]
\( X^m \) is orthogonal \((X^m \partial_\pm X^m = 0)\) to both \(X_+^m\) and \(X_-^m\)
remaining \(SO(3)\) invariant quantity is scalar product
\[ \partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi \]
then \(\partial_+ \partial_- \varphi + \frac{\mu^2}{2} \sin 2\varphi = 0\)
following from \textbf{sine-Gordon action} (Pohlmeyer, 1976)
\[ \widetilde{L} = \partial_+ \varphi \partial_- \varphi + \frac{\mu^2}{2} \cos 2\varphi \]

2d Lorentz invariant despite explicit constraints
Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating folded string on \(S^2\)
Analogous construction for \(S^3\) model gives
\textbf{Complex sine-Gordon model} (Pohlmeyer; Lund, Regge 76)
\[ \widetilde{L} = \partial_+ \varphi \partial_- \varphi + \cot^2 \varphi \partial_+ \theta \partial_- \theta + \frac{\mu^2}{2} \cos 2\varphi \]
“String on $R_t \times S^n$” interpretation

conformal gauge plus $t = \mu \tau$ to fix conformal diffeomorphisms:

\[ \partial_\pm X^m \partial_\pm X^m = \mu^2 \]

are Virasoro constraints

String on $AdS_n \times S^1_\psi$ with $\psi = \mu \tau$

e.g. reduced theory for $AdS_3 \times S^1$

\[
\tilde{L} = \partial_+ \phi \partial_- \phi + \coth^2 \varphi \partial_+ \chi \partial_- \chi - \frac{\mu^2}{2} \cosh 2\phi
\]

- resulting Lagrangian turns out to be 2d Lorentz invariant
- reduced theory is equivalent to the original theory as integrable system: the respective Lax pairs are gauge-equivalent
- formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge
- In general reduced theory cannot be quantum-equivalent to the original one (e.g., conformal symmetry was assumed in the reduction procedure)
PR for bosonic $F/G$-coset model

$G/H$ gauged WZW model modified by integrable potential

symmetric space condition ($\mathfrak{f}$, $\mathfrak{g}$ are Lie algebras of $F$ and $G$)

$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g} , \quad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} , \quad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p} , \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$

Lagrangian:

$$L = -\text{Tr}(P_+ P_-) , \quad P_\pm = (f^{-1}\partial_\pm f)_p ,$$

$$J = f^{-1}df = A + P , \quad A = J_\mathfrak{g} \in \mathfrak{g} , \quad P = J_\mathfrak{p} \in \mathfrak{p}$$

Let $J = A + P$ be fundamental variables, not $f$

$$D_+ P_- = 0 , \quad D_- P_+ = 0 , \quad D = d + [A, \ ] \quad -\text{EOM}$$

$$D_- P_+ - D_+ P_- + [P_+, P_-] + \mathcal{F}_{+-} = 0 \quad -\text{Maurer-Cartan}$$

$$\text{Tr}(P_+ P_+) = -\mu^2 , \quad \text{Tr}(P_- P_-) = -\mu^2 \quad -\text{Virasoro}$$

*Main idea:* first solve EOM and Virasoro and then MC
gauge fixing that solves the first Virasoro constraint

\[ P_+ = \mu T = \text{const} \quad , \quad T \in \mathfrak{p} = \mathfrak{f} \otimes \mathfrak{g}, \]

\( \mathfrak{h} \) is a centraliser of \( T \) in \( \mathfrak{g} \): \( [\mathfrak{h}, T] = 0 \)

second Virasoro constraint is solved by

\[ P_- = \mu g^{-1}Tg \quad , \quad g \in G \]

EOM:

\[ A_- = A_- \quad , \quad A_+ = g^{-1}\partial_+ g + g^{-1}A_+ g \quad , \quad A_\pm \in \mathfrak{h} \]
solved EOM’s and Virasoro constraints in terms of

\( G \)-valued \( g \); \( \mathfrak{h} \)-valued \( A_+, A_- \), \( [T, A_\pm] = 0 \)

remaining equations on \( g, A_\pm \) follow from

\( G/H \) gWZW action with potential:

\[
L = - \frac{1}{2} \text{Tr}(g^{-1}\partial_+ gg^{-1}\partial_- g) + \text{WZ term} \\
- \text{Tr}(A_+ \partial_- gg^{-1} - A_- g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-) \\
- \mu^2 \text{Tr}(T g^{-1} T g)
\]
Pohlmeyer-reduced theory for $F/G$ coset sigma model
(as first proposed by Bakas, Park, Shin 95)
and thus also for strings on $R^t \times F/G$ or $F/G \times S^1_\psi$
integrable potential: relation at the level of Lax pairs
special case of non-abelian Toda theory:
“symmetric space Sine-Gordon model”
What to do with $A_+, A_-$: integrate out or gauge-fix
Reduced equation of motion in the “on-shell” gauge $A_\pm = 0$:
\[
\partial_-(g^{-1}\partial_+ g) - \mu^2[T, g^{-1}T g] = 0,
\]
\[
(g^{-1}\partial_+ g)_h = 0,
\]
\[
(\partial_- gg^{-1})_h = 0.
\]

$F/G = SO(n+1)/SO(n) = S^n$: $G/H = SO(n)/SO(n-1)$
Linearising around the vacuum $g = 1$
\[
\partial_+ \partial_- k_\ell + \mu^2 k_\ell + O(k_\ell^2) = 0
\]

massive spectrum: non-trivial S-matrix with $H$ global symmetry
F/G = SO(n + 1)/SO(n) = S^n:
parametrization of g in Euler angles
\[ g = e^{T_{n-2}\theta_{n-2}} e^{T_1 \theta_1} e^{2T \varphi} e^{T_1 \theta_1} e^{T_{n-2}\theta_{n-2}} \]
and integrating out \( H = SO(n-1) \) gauge field \( A_\pm \)
leads to reduced theory that generalizes SG and CSG

\[ \tilde{L} = \partial_+ \varphi \partial_- \varphi + G_{pq}(\varphi, \theta) \partial_+ \theta^p \partial_- \theta^q + \frac{\mu^2}{2} \cos 2\varphi \]

Bosonic strings on \( AdS_n \times S^n \)

\[ L = \text{Tr}(P_+^A P_-^A) - \text{Tr}(P_+^S P_-^S) \]
\[ \text{Tr}(P_\pm^S P_\pm^S) - \text{Tr}(P_\pm^A P_\pm^A) = 0 \]
\[ \text{Tr}(P_\pm^S P_\pm^S) = \text{Tr}(P_\pm^A P_\pm^A) = -\mu^2 \]

get direct sum of reduced systems for \( S^n \) and \( AdS_n \)
linked by Virasoro, i.e. common \( \mu \)
e.g. for \( F/G = AdS_2 \times S^2 \):

\[ \tilde{L} = \partial_+ \varphi \partial_- \varphi + \partial_+ \phi \partial_- \phi + \frac{\mu^2}{2} (\cos 2\varphi - \cosh 2\phi) \]
**AdS$_5 \times S^5$ superstring sigma-model**

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

supercoset GS sigma model (Metsaev, AT 98)

$\hat{F} = \frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)}$

basic superalgebra $\hat{f} = psu(2, 2|4)$

bosonic part $f = su(2, 2) \oplus su(4) \cong so(2, 4) \oplus so(6)$

admits $Z_4$-grading: (Berkovits, Bershadsky, et al 89)

$$\hat{f} = f_0 \oplus f_1 \oplus f_2 \oplus f_3 , \quad [f_i, f_j] \subset f_{i+j \mod 4}$$

$f_0 = g = sp(2, 2) \oplus sp(4)$

current ($J = f^{-1} \partial_a f, \ f \in \hat{F}$) decomposes as

$$J_a = f^{-1} \partial_a f = A_a + Q_{1a} + P_a + Q_{2a}$$

$A \in f_0, \quad Q_1 \in f_1, \quad P \in f_2, \quad Q_2 \in f_3$ .
GS Lagrangian in terms of $8 \times 8$ supermatrices:

$$L_{GS} = \frac{1}{2} \text{STr} \left( \sqrt{-g} g^{ab} P_a P_b + \varepsilon^{ab} Q_{1a} Q_{2b} \right),$$

very simple structure – but not standard coset model:
fermionic currents in WZ term only
\textbf{conformal gauge:} $\sqrt{-g} g^{ab} = \eta^{ab}$

In terms of current $J_\pm = \mathcal{A}_\pm + P_\pm + Q_{1\pm} + Q_{2\pm}$

\textbf{EOM}:
\begin{align*}
\partial_+ P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] &= 0, \\
\partial_- P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] &= 0, \\
[P_+, Q_{1-}] &= 0, \quad [P_-, Q_{2+}] = 0.
\end{align*}

\textbf{Virasoro}:
\begin{align*}
\text{STr}(P_+ P_+) &= 0, \quad \text{STr}(P_- P_-) = 0
\end{align*}

\textbf{MC}:
\begin{align*}
\partial_- J_+ - \partial_+ J_- + [J_- , J_+] &= 0.
\end{align*}

\textbf{PR procedure:} solve first EOM and Virasoro

\textit{$\kappa$-symmetry gauge condition:} $Q_{1-} = 0$, $Q_{2+} = 0$
\[ P_+ = \mu T, \quad T = \frac{i}{2} \text{diag}(1, 1, -1, -1|1, 1, -1, -1) \]

\[ P_- = \mu g^{-1} T g, \quad A_+ = g^{-1} \partial_+ g + g^{-1} A_+ g, \quad A_- = A_- \]

\( T \) defines \( \mathfrak{h} \): \( [\mathfrak{h}, T] = 0 \):
\[ \mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \]

new parametrisation: \( G = Sp(2, 2) \times Sp(4) \)-valued field \( g \)

and \( \mathfrak{h} \)-valued field \( A_\pm \)

define
\[ \Psi_R = Q_{1+}, \quad \Psi_L = gQ_{2-}g^{-1} \]
PR theory for $AdS_5 \times S^5$ superstring

fermionic generalization of “gWZW+ potential” theory for
\[ \frac{G}{H} = \frac{Sp(2,2)}{SU(2) \times SU(2)} \times \frac{Sp(4)}{SU(2) \times SU(2)} \]

\[ L = L_{gWZW}(g, A_+, A_-) + \mu^2 \text{STr}(g^{-1}TgT) + \text{STr} (\Psi_L TD_+ \Psi_L + \Psi_R TD_- \Psi_R) + \mu \text{STr} (g^{-1}\Psi_L g\Psi_R) \]

direct sum of bosonic PR theories for $AdS_5$ and $S^5$
“glued together” by components of fermions
\[ L = \tilde{L}_{S^5}(g, A_+, A_-) + \tilde{L}_{AdS_5}(g, A_+, A_-) + \psi_L D_+ \psi_L + \psi_R D_+ \psi_R + \mu \text{ (interaction terms)} \]

all gauge symmetries fixed; standard kin. terms (cf. GS action)
UV finite as a quantum theory (Roiban, AT, to appear)
• Bosonic gWZW model coupled to fermions interacting minimally and through the “Yukawa term”
• 8 real bosonic and 16 real fermionic independent variables
• 2d Lorentz invariant with $\Psi_R, \Psi_L$ as 2d Majorana spinors
• 2d supersymmetry? yes, at the linearised level, and yes in $AdS_2 \times S^2$ case: $n = 2$ super sine-Gordon
• $\mu$-dependent interaction terms are equal to original GS Lagrangian; gWZW produces MC eq.: path integral derivation via change from fields to currents?
• quadratic in fermions (like susy version of gWZW); integrating out $A_\pm$ gives quartic fermionic terms (reflecting curvature)
• linearisation of EOM in the gauge $A_\pm = 0$ around $g = 1$ describes 8+8 massive bosonic and fermionic d.o.f. with mass $\mu$: same as in BMN limit
• symmetry of resulting relativistic S-matrix: $H = [SU(2)]^4$ – same as bosonic part of magnon S-matrix symmetry $[PSU(2|2)]^2$
Example: superstring on $AdS_2 \times S^2$

Explicit parametrisation:

\[ T = \frac{1}{2} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \]

\[ g = \begin{pmatrix} \cosh \phi & \sinh \phi & 0 & 0 \\ \sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & \cos \varphi & i \sin \varphi \\ 0 & 0 & i \sin \varphi & \cos \varphi \end{pmatrix}. \]

\[ \Psi_R = \begin{pmatrix} 0 & 0 & 0 & i \gamma \\ 0 & 0 & -\beta & 0 \\ 0 & i \beta & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}, \quad \Psi_L = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & -i \nu & 0 \\ 0 & \nu & 0 & 0 \\ i \rho & 0 & 0 & 0 \end{pmatrix}. \]
PR Lagrangian: same as $n = 2$ supersymmetric sine-Gordon!

$$\tilde{\mathcal{L}} = \partial_+ \varphi \partial_- \varphi + \partial_+ \phi \partial_- \phi + \frac{\mu^2}{2} (\cos 2\varphi - \cosh 2\phi)$$

$$+ \beta \partial_- \beta + \gamma \partial_- \gamma + \nu \partial_+ \nu + \rho \partial_+ \rho$$

$$- 2\mu \left[ \cosh \phi \cos \varphi \left( \beta \nu + \gamma \rho \right) + \sinh \phi \sin \varphi \left( \beta \rho - \gamma \nu \right) \right].$$

Indeed, equivalent to

$$\tilde{\mathcal{L}} = \partial_+ \Phi \partial_- \Phi^* - |W'(\Phi)|^2$$

$$+ \psi^*_L \partial_+ \psi_L + \psi^*_R \partial_- \psi_R + \left[ W''(\Phi) \psi_L \psi_R + W'''(\Phi^*) \psi^*_L \psi^*_R \right].$$

Bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu \cos \Phi, \quad |W'(\Phi)|^2 = \frac{\mu^2}{2} (\cosh 2\phi - \cos 2\varphi).$$

$$\psi_L = \nu + i\rho, \quad \psi_R = -\beta + i\gamma,$$
Open questions

- Quantum equivalence of reduced theory and GS theory?
- Path integral argument of equivalence?
  Potential term is original action
  \[ \text{Tr}(P_+ P_-) = \mu^2 \text{Tr}(T g^{-1} T g) \]
  gWZW should come from change of variables
- Indication of equivalence: semiclassical expansion– near analog of spinning string in \( AdS_5 \times S^5 \) leads to the same 1-loop partition function (Roiban, AT 08, to appear)
- Tree-level S-matrix for elementary excitations? Relation to magnon S-matrix in gauge theory Bethe Ansatz?
- Better understanding of relationship between original and reduced system: symmetries, vacua, charges, etc.; which observables can be related?
Conclusions

GS action in $AdS_5 \times S^5$: well-defined quantum theory
basis for perturbative $\frac{1}{\sqrt{\lambda}}$ computations
provides data for construction of all-order BA for the spectrum

“Pohlmeyer reduction”: most promising approach
towards solution of $AdS_5 \times S^5$ GS superstring
Uncovers remarkable connection to a UV finite fermionic
(2d supersymmetric?)
integrable deformation of a gWZW model
solvable by Bethe Ansatz?
same as BA on gauge theory side?